HW 11 - SOLUTIONS

PROBLEM 1

(a) $$U^\varepsilon(t,x) = \bar{U}^\varepsilon\left(t, \frac{x - s^\varepsilon(t)}{\varepsilon}\right)$$

$$U^\varepsilon_t = \bar{U}^\varepsilon_t + \bar{U}^\varepsilon_y\left(-\frac{s'(t)}{\varepsilon}\right)$$

$$U^\varepsilon_x = \bar{U}^\varepsilon_y\left(\frac{1}{\varepsilon}\right), \quad U^\varepsilon_{xx} = \bar{U}^\varepsilon_{yy}\left(\frac{1}{\varepsilon^2}\right)$$

SO THE PDE BECOMES

$$\varepsilon^2 \bar{U}^\varepsilon_t + \cancel{\varepsilon^2} \bar{U}^\varepsilon_y\left(-\frac{s'(t)}{\cancel{\varepsilon}}\right) - \cancel{\varepsilon^2} \bar{U}^\varepsilon_{yy}\left(\frac{1}{\cancel{\varepsilon^2}}\right) + (f(x))^2 \sin(2\bar{U}^\varepsilon) = 0$$

$$\varepsilon^2 \bar{U}^\varepsilon_t - \varepsilon s'(t) \bar{U}^\varepsilon_y - \bar{U}^\varepsilon_{yy} + \left(f(s + \varepsilon y)\right)^2 \sin(2\bar{U}^\varepsilon) = 0$$

$$(x = s + \varepsilon y)$$

(b) ANSATZ $$\bar{U}^\varepsilon = \bar{U}^0 + \varepsilon \bar{U}^1 + \dots$$

$$s^\varepsilon = s^0 + \varepsilon s^1 + \dots$$

$$\varepsilon^2 \bar{U}^0_t + \varepsilon^2 \bar{U}^1_t - \varepsilon s_0' \bar{U}^0_y - \bar{U}^0_{yy} - \varepsilon \bar{U}^1_{yy}$$

$$+ \left(f(s_0) + (\varepsilon s^1 + \varepsilon y) f'(s_0)\right)^2 \left(\sin(2\bar{U}^0) + 2\varepsilon \bar{U}^1 \cos(2\bar{U}^0)\right) =$$

$$\sin(2a + h) = \sin(2a) + h\cos(2a)$$

$$(f(s_0))^2 \sin(2\bar{U}^0) + 2\bar{U}^1 (f(s_0))^2 \cos(2\bar{U}^0)\,\varepsilon + 2 f(s_0)(s^1 + y) f'(s_0) \sin(2\bar{U}^0)\,\varepsilon$$

$O(1)$ $$-\overline{U}^0_{YY} + (f(s_0))^2 \sin(2\overline{U}^0) = 0 \qquad (*)$$

$O(\varepsilon)$ $$-s_0'\,\overline{U}^0_Y - \overline{U}^1_{YY} + 2\overline{U}^1 (f(s_0))^2 \cos(2\overline{U}_0) + 2(s^1+Y) f(s_0) f'(s_0) \sin(2\overline{U}^0) = 0$$

(c) MULTIPLY $O(\varepsilon)$ BY $\overline{U}^0_Y$ AND INTEGRATE W.R.T. Y ON $\mathbb{R}$:

$$\int_{-\infty}^{\infty} -s_0'(\overline{U}^0_Y)^2 - \overline{U}^1_{YY}\,\overline{U}^0_Y + 2\overline{U}^1 (f(s_0))^2 \cos(2\overline{U}_0)\,\overline{U}^0_Y + 2(s^1+Y) f(s_0) f'(s_0) \sin(2\overline{U}^0)\,\overline{U}^0_Y \, dY = 0$$

$$\int_{-\infty}^{\infty} A + B + C + D \, dY = 0$$

A JUST BECOMES $-(s_0') \int_{-\infty}^{\infty} (\overline{U}^0_Y)^2$

INTEGRATING B BY PARTS, WE GET:

$$\int_{-\infty}^{\infty} -\overline{U}^1_{YY}\,\overline{U}^0_Y = \int_{-\infty}^{\infty} \overline{U}^1_Y (\overline{U}^0_{YY}) \qquad \text{BY } (*)$$

$$= \int_{-\infty}^{\infty} \overline{U}^1_Y \left((f(s_0))^2 \sin(2\overline{U}^0) \right) dY$$

$$= (f(s_0))^2 \int_{-\infty}^{\infty} \overline{U}^1_Y \sin(2\overline{U}^0)$$

AS FOR C, NOTICE THAT:

$$\int_{-\infty}^{\infty} 2\overline{U}^1 f(s_0) \cos(2\overline{U}^0)\,\overline{U}^0_Y = (f(s_0))^2 \int_{-\infty}^{\infty} \overline{U}^1 \left[\sin(2\overline{U}^0)\right]_Y dY$$

$$= -(f(s_0))^2 \int_{-\infty}^{\infty} \overline{U}^{2}_{Y} \, sw(2\overline{U}^0)\, dy \quad (IBP) = -B$$

THEREFORE WE GET:

$$\int_{-\infty}^{\infty} A + \cancel{B} + \underbrace{C}_{-\cancel{B}} + D \, dy = 0$$

SO WE'RE LEFT WITH:

$$0 = \int_{-\infty}^{\infty} (-s_0')(\overline{U}^0_Y)^2 + 2(s^2+y)\, f(s_0)\, f''(s_0)\, \underbrace{sw(2\overline{U}^0)}_{\frac{\overline{U}^0_{YY}}{(f(s_0))^2}}\, \overline{U}^0_Y \, dy$$

$$= -(s_0') \int_{-\infty}^{\infty} (\overline{U}^0_Y)^2 + \int_{-\infty}^{\infty} (s^2+y)\, \cancel{f(s_0)}\, f''(s_0) \left(\frac{1}{(f(s_0))^{\cancel{2}}}\right) dy$$

$$\times \underbrace{2\,\overline{U}^0_{YY}\,\overline{U}^0_Y}_{[(\overline{U}^0_Y)^2]_Y} \, dy$$

IBP

$$= -(s_0') \int_{-\infty}^{\infty} (\overline{U}^0_Y)^2 - \frac{f''(s_0)}{f(s_0)} \int_{-\infty}^{\infty} (\overline{U}^0_Y)^2 \, dy$$

$$0 = \left[-(s_0') - \frac{f''(s_0)}{f(s_0)}\right] \left(\cancel{\int_{-\infty}^{\infty} (\overline{U}^0_Y)^2 \, dy} \right)$$

$$\Rightarrow \quad -(s_0)' - \frac{f''(s_0)}{f(s_0)} = 0$$

$$\Rightarrow \quad \boxed{(s_0)' = -\frac{f''(s_0)}{f(s_0)}}$$

PROBLEM 2

(a) $$U_t^\varepsilon + U^\varepsilon U_x^\varepsilon + \varepsilon U_{xx}^\varepsilon = 0$$

ANSATZ $$U^\varepsilon = U^0 + \varepsilon U^1 + \dots$$

$$U_t^0 + \varepsilon U_t^1 + (U^0 + \varepsilon U^1)(U_x^0 + \varepsilon U_x^1) + \varepsilon U_{xx}^0 + \varepsilon^2 U_{xx}^1 = 0$$

$$U_t^0 + \varepsilon U_t^1 + U^0 U_x^0 + \varepsilon \dots + \varepsilon U_{xx}^0 + \varepsilon^2 U_{xx}^1 = 0$$

$O(1)$ $$\boxed{U_t^0 + U^0 U_x^0 = 0}$$

(b) $$y = \frac{x - s(t)}{\varepsilon}, \quad U^\varepsilon(t,x) = \overline{U}^\varepsilon\left(t, \frac{x-s(t)}{\varepsilon}\right)$$

$$U_t^\varepsilon = \overline{U}_t^\varepsilon + \overline{U}_y^\varepsilon \left(-\frac{s'(t)}{\varepsilon}\right)$$

$$U_x^\varepsilon = \overline{U}_y^\varepsilon \left(\frac{1}{\varepsilon}\right)$$

$$U_{xx}^\varepsilon = \overline{U}_{yy}^\varepsilon \left(\frac{1}{\varepsilon^2}\right)$$

HENCE WE GET:

$$\overline{U}_t^\varepsilon + \left(-\frac{s'(t)}{\varepsilon}\right)\overline{U}_y^\varepsilon + \overline{U}^\varepsilon \overline{U}_y^\varepsilon\left(\frac{1}{\varepsilon}\right) + \cancel{\varepsilon}\, \overline{U}_{yy}^\varepsilon\left(\frac{1}{\varepsilon^{\cancel{2}}}\right) = 0$$

ANSATZ $$\overline{U}^\varepsilon = \overline{U}^0 + \varepsilon \overline{U}^1 + \dots$$

$$\overline{U}_t^0 + \varepsilon \overline{U}_t^1 - \frac{s'(t)}{\varepsilon}\overline{U}_y^0 - s'(t)\overline{U}_y^1 + \overline{U}^0 \overline{U}_y^0\left(\frac{1}{\varepsilon}\right) + \overline{U}_{yy}^0\left(\frac{1}{\varepsilon}\right) = 0$$

$O\left(\frac{1}{\varepsilon}\right)$ $$\boxed{-s'(t)\overline{U}_y^0 + \overline{U}^0 \overline{U}_y^0 + \overline{U}_{yy}^0 = 0}$$

(c) INTEGRATING IN Y FROM $-\infty$ TO ∞, WE GET

$$\int_{-\infty}^{\infty} -s'(t)\,\overline{U^\circ_Y} + \underbrace{\overline{U^\circ}\,\overline{U^\circ_Y}}_{\frac{1}{2}\left[(\overline{U^\circ})^2\right]_Y} + \overline{U^\circ_{YY}}\; dY = 0$$

$$-s'(t)\left[\overline{U^\circ}\right]_{Y=-\infty}^{Y=\infty} + \left[\frac{(\overline{U^\circ})^2}{2}\right]_{Y=-\infty}^{Y=\infty} + \underbrace{\left[\overline{U^\circ_Y}\right]_{Y=-\infty}^{Y=\infty}}_{0\text{, BY ASSUMPTION}} = 0$$

$$-s'(t)\left(\lim_{Y\to\infty}\overline{U^\circ}(Y) - \lim_{Y\to-\infty}\overline{U^\circ}(Y)\right)$$

$$+\frac{1}{2}\left(\lim_{Y\to\infty}\left(\overline{U^\circ}(Y)\right)^2 - \lim_{Y\to-\infty}\left(\overline{U^\circ}(Y)\right)^2\right) = 0$$

$$-s'(t)\left((U^\circ)^+ - (U^\circ)^-\right) + \frac{1}{2}\left(\left[(U^\circ)^+\right]^2 - \left[(U^\circ)^-\right]^2\right) = 0$$

SOLVING FOR $s'(t)$, WE GET:

$$s'(t) = \frac{\frac{1}{2}\left(\left[(U^\circ)^+\right]^2 - \left[(U^\circ)^-\right]^2\right)}{(U^\circ)^+ - (U^\circ)^-}$$

$$= \frac{1}{2}\frac{\cancel{\left[(U^\circ)^+ - (U^\circ)^-\right]}\left((U^\circ)^+ + (U^\circ)^-\right)}{\cancel{(U^\circ)^+ - (U^\circ)^-}}$$

$$= \frac{1}{2}\left((U^\circ)^+ + (U^\circ)^-\right)$$

$$\boxed{s'(t) = \frac{(U^\circ)^+ + (U^\circ)^-}{2}}$$