Math 379 – Homework 11 (the last one!)

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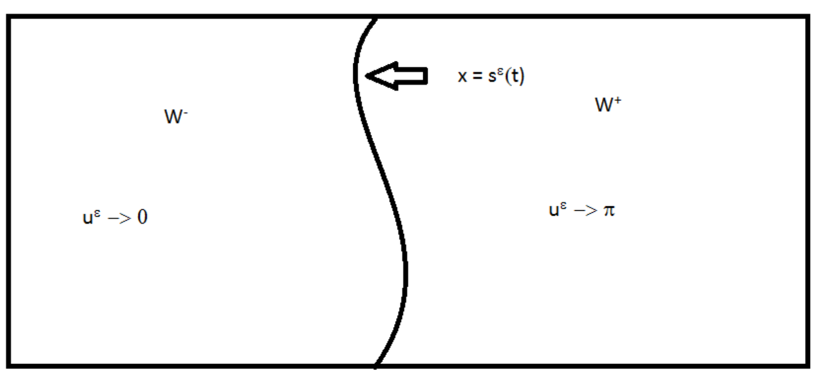
Problem 1: Consider the following PDE:

$$\epsilon^2 u_t^\epsilon - \epsilon^2 u_{xx}^\epsilon + (f(x))^2 \sin(2u^\epsilon) = 0,$$

where $u^\epsilon = u^\epsilon(t,x), x \in \mathbb{R}$, $f(x) > 0$. Here you should think of $\sin(2u^\epsilon)$ as being our $\Phi'(u^\epsilon)$. Similar to the situation in Example 5 (A Singular Variational Problem), we expect that there are two regions $W^\pm$ separated by a curve $x = s^\epsilon(t)$ such that

$$u^\epsilon(t,x) \to \begin{cases} 0 \text{ if } x < s^\epsilon(t) \\ \pi \text{ if } x > s^\epsilon(t) \end{cases}$$

as in the following picture:



Goal: We would like to find a differential equation for $s^0(t)$.

(a) Let $u^\epsilon(t,x) = \bar{u}^\epsilon\left(t, \frac{x-s^\epsilon(t)}{\epsilon}\right)$, where $\bar{u}^\epsilon = \bar{u}^\epsilon(t,y)$. Then rewrite the above PDE in terms of $\bar{u}^\epsilon$.

(b) Now apply the Ansätze $\bar{u}^\epsilon = \bar{u}^0 + \epsilon\bar{u}^1 + \cdots$ and $s^\epsilon = s^0 + \epsilon s^1 + \cdots$, to the PDE in (a), where $\bar{u}^k = \bar{u}^k(t,y)$ and $s^k = s^k(t)$. You may need to Taylor expand the $\sin(2\bar{u}^\epsilon)$ term and the $f(x) = f(s+\epsilon y) = f(s_0 + \epsilon(s_1 + y))$ term. What are the $O(1)$ and $O(\epsilon)$ terms?

(c) Multiply the equation you found for the $O(\epsilon)-$term by $\bar{u}^0_y$ and integrate with respect to y on $\mathbb{R}$. You should get four terms. For term involving $\bar{u}^1_{yy}$, integrate by parts and use the $O(1)-$ term. For the term that has a $\cos(2\bar{u}^0)\bar{u}^0_y$ (recognize this as a derivative), integrate by parts. You should see some cancellation happening now when you add the 4 terms. For the expression that involves $\sin(2\bar{u}^0)$, don't integrate by parts yet; first use your $O(1)$ term to get rid of the $\sin -$term and then integrate by parts.

Finally, show that s_0 must satisfy the differential equation:

$$s_0'(t) = -\frac{f'(s_0)}{f(s_0)}.$$

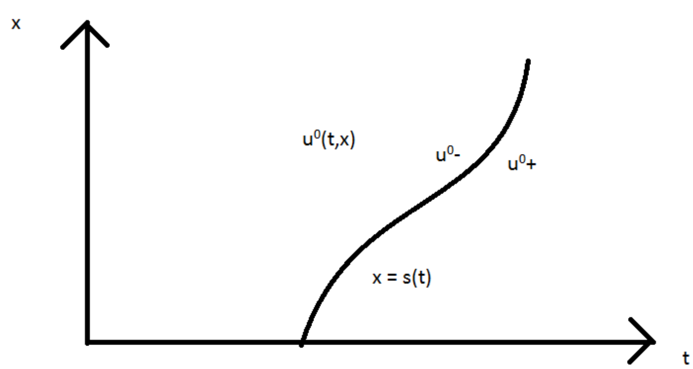
Here $s_0' = \frac{ds^0}{dt}$ and $f' = \frac{df}{ds}$.

(TURN PAGE for Problem 2)

Problem 2: Consider Burgers' equation

$$u_t^\epsilon + u^\epsilon u_x^\epsilon + \epsilon u_{xx}^\epsilon = 0 \tag{B}$$

where $u^\epsilon = u^\epsilon(t, x)$ with $x \in \mathbb{R}$. From PDE$-$theory, it turns out that u^0 forms a 'shock' along a curve $x = s(t)$, that is u^0 has a jump discontinuity, where u_0 jumps from values u_0^- to u_0^+), as in the following picture:



Goal: Find a differential equation for s

(a) [Outer Solution, far from $s(t)$]

Apply the Ansatz $u^\epsilon = u^0 + \epsilon u^1 + \cdots$ and show that u^0 satisfies the PDE

$$u_t^0 + u^0 u_x^0 = 0$$

(b) [Inner Solution, on $s(t)$]

Assume that u^0 is discontinuous along the curve $x = s(t)$. Let $y = \frac{x - s(t)}{\epsilon}$ and $u^\epsilon(t, x) = \bar{u}^\epsilon(t, y)$.

Rewrite (B) in terms of $\bar{u}_\epsilon$, apply the Ansatz

$$\bar{u^\epsilon} = \bar{u}^0 + \epsilon \bar{u}^1 + \cdots$$

to (B), compare the $O(\frac{1}{\epsilon})$ terms, and find a PDE for $\bar{u}^0$ (no need to Ansatz s here!)

(c) [Matching] Integrate the equation in (b) with respect to y from $-\infty$ to ∞. You may assume that:

$$\lim_{y\to\pm\infty} \bar{u}^0_y(y) = 0.$$

The matching assumption here becomes:

$$\begin{cases} \lim_{y\to\infty} \bar{u}^0(y) = \lim_{x\to(s(t))^+} u^0 := u_0^+ \\ \lim_{y\to-\infty} \bar{u}^0(y) = \lim_{x\to(s(t))^-} u^0 := u_0^- \end{cases}$$

Use the matching assumption to find that s solves the following ODE:

$$s'(t) = \frac{u_0^- + u_0^+}{2}.$$

Notice an averaging-phenomenon happening here, similar to Example 3: "An internal layer."