HW 10 - SOLUTIONS

PROBLEM 1 WE HAVE $e = r_0 (v_0)^2 - 1$ (BY DEF OF e), SO SOLVING FOR $(v_0)^2$,

WE GET: $$(v_0)^2 = \frac{1}{r_0}(e+1) = \frac{1}{r_0}\left(\frac{1}{\cos(\alpha)} + 1\right) = \frac{1}{r_0}\left(\frac{1+\cos(\alpha)}{\cos(\alpha)}\right)$$

NOW $$\frac{(v_\infty)^2}{2} = \frac{(v_0)^2}{2} - \frac{1}{r_0} = \frac{1}{2r_0}\left(\frac{1+\cos(\alpha)}{\cos(\alpha)}\right) - \frac{1}{r_0}$$

$$= \frac{1+\cos(\alpha) - 2\cos(\alpha)}{2 r_0 \cos(\alpha)} = \frac{1-\cos(\alpha)}{2 r_0 \cos(\alpha)}$$

SO $(v_\infty)^2 = \dfrac{1-\cos(\alpha)}{r_0 \cos(\alpha)}$, BUT THEN

BUT $$\frac{\tan(\alpha)}{\ell} = \tan(\alpha) \frac{\sin(\alpha)}{(\cos(\alpha)+1) r_0}$$

$$= \frac{\sin^2(\alpha)}{\cos(\alpha)(\cos(\alpha)+1) r_0}$$

$$= \frac{1-\cos^2(\alpha)}{\cos(\alpha)(\cos(\alpha)+1) r_0}$$

$$= \frac{(1-\cos(\alpha))\cancel{(1+\cos(\alpha))}}{\cos(\alpha)\cancel{(\cos(\alpha)+1)} r_0}$$

$$= \frac{1-\cos(\alpha)}{\cos(\alpha) r_0}$$

$$= (v_\infty)^2$$

SO $$\boxed{(v_\infty)^2 = \frac{\tan(\alpha)}{\ell}}$$

PROBLEM 2 (a) OUTER SOLUTION [NEAR $x=0$]

ANSATZ $\quad p^{\varepsilon}(x) = p^{0}(x) + \varepsilon p^{1}(x) + \cdots$

$$-\varepsilon\Big((h^{3})\big(p^{0}(x)+\varepsilon p^{1}(x)\big)\big(p^{0}_{x}(x)+\varepsilon p^{1}_{x}(x)\big)\Big)_{x}$$

$$= \Big(\big(p^{0}(x)+\varepsilon p^{1}(x)\big)h\Big)_{x}$$

$$-\varepsilon\,(\ldots) \qquad = (p^{0}(x)h)_{x} + \varepsilon\,(p^{1}(x)h)_{x}$$

$O(1)$ - TERMS $\qquad (p^{0}(x)h)_{x} = 0$

$$\Rightarrow \quad p^{0}(x)\,h(x) = C \qquad (*)$$

IMPOSE $\quad p^{0}(0) = 1$

USE $(*)$ WITH $x=0$ TO GET $\quad \underbrace{p^{0}(0)}_{1}\,h(0) = C \Rightarrow C = h(0)$

AND SO $(*)$ BECOMES $\quad p^{0}(x)\,h(x) = h(0)$

$$\boxed{p^{0}(x) = \frac{h(0)}{h(x)}}$$

(b) INNER SOLUTION [NEAR $x=1$]

$$y = \frac{x-1}{\varepsilon}, \quad \bar{p}_{\varepsilon}(y) = p_{\varepsilon}(x), \quad \bar{h}(y) = h(x)$$

$$p^{\varepsilon}_{x} = \frac{dp^{\varepsilon}}{dx} = \frac{d\bar{p}^{\varepsilon}}{dy}\,\frac{dy}{dx} = \bar{p}^{\varepsilon}_{y}\left(\frac{1}{\varepsilon}\right)$$

AND SIMILARLY $(\cdots)_x = \frac{1}{\varepsilon}(\cdots)_y$

AND SO $-\varepsilon\left((h^3)\, p^\varepsilon p^\varepsilon_x\right)_x = (p^\varepsilon h)_x$ BECOMES

$$-\cancel{\varepsilon}\,\cancel{\frac{1}{\varepsilon}}\left[(\bar h^3)\,\bar p^\varepsilon \frac{1}{\varepsilon}\bar p^\varepsilon_y\right]_y = (\bar p^\varepsilon \bar h)_y \cancel{\frac{1}{\varepsilon}}$$

$$-\left((\bar h^3)\,\bar p^\varepsilon\, \bar p^\varepsilon_y\right)_y = (\bar p^\varepsilon \bar h)_y$$

ANSATZ $\bar p^\varepsilon = \bar p^0 + \varepsilon \bar p^1 + \dots$

$\bar h(y) = h(1+\varepsilon y) = h(1) + \varepsilon y h'(1) + \dots$

$$-\left((h(1)+\varepsilon y h'(1))^3 (\bar p^0+\varepsilon \bar p^1)(\bar p^0_y + \varepsilon \bar p^1_y)\right)_y = \left((\bar p^0+\varepsilon \bar p^1)(h(1)+\varepsilon y h'(1))\right)_y$$

$$-\left((h(1))\, \bar p^0\, \bar p^0_y\right)_y + \varepsilon \cdots = (\bar p^0 h(1))_y + \varepsilon \cdots$$

O(1)-TERMS $-\left((h(1))^3\, \bar p^0\, \bar p^0_y\right)_y = (\bar p^0 h(1))_y$

$$-\underbrace{(h(1))^3}_{1}(\bar p^0 \bar p^0_y)_y = \underbrace{h(1)}_{1}(\bar p^0_y)$$

$$-(\bar p^0\bar p^0_y)_y = \bar p^0_y$$

$$\Rightarrow -(\bar p^0 \bar p^0_y) = \bar p^0 + A$$

$$\Rightarrow \boxed{\bar p^0 \bar p^0_y + \bar p^0 = -A}$$

↳ LET'S SOLVE THIS! (SEE NEXT PAGE)

HOW TO SOLVE $yy' + y = -A \qquad y = y(t)$

$$yy' = -A - y$$

$$y' = \frac{-A-y}{y}$$

$$\frac{dy}{dt} = -\left(\frac{A+y}{y}\right)$$

$$\left(\frac{y}{A+y}\right) dy = -dt$$

$$\int \frac{y}{A+y}\, dy = \int -dt$$

$$\frac{y}{A+y} = \frac{A+y}{A+y} - \frac{A}{A+y} = 1 - \frac{A}{A+y}$$

$$y - A\ln|A+y| = -t + C$$

$$\Rightarrow -t = y - A\ln|A+y| - C$$

NOW REPLACING t WITH Y AND y WITH $\bar{p}^\circ$, WE GET.

$$-Y = \bar{p}^\circ - A\ln|A+\bar{p}^\circ| - C$$

IMPOSE $\bar{p}^\circ(0) = 1$ TO GET:

$$-0 = \bar{p}^\circ(0) - A\ln|A+\bar{p}^\circ(0)| - C$$

$$0 = 1 - A\ln|A+1| - C \Rightarrow C = 1 - A\ln|A+1|$$

AND SO WE GET:

$$-Y = \bar{p}^{o} - A \ln|A+\bar{p}^{o}| - 1 + A\ln|A+1|$$

$$-Y = \bar{p}^{o} - A\left(\ln|A+\bar{p}^{o}| - \ln|A+1|\right) - 1$$

$$-Y = \bar{p}^{o} - A\left(\ln\left|\frac{A+\bar{p}^{o}}{A+1}\right|\right) - 1 \quad (I)$$

(c) MATCHING

$$\lim_{X\to 1} p^{o}(X) = \underbrace{\lim_{Y\to-\infty} \bar{p}^{o}(Y)}$$

$$\lim_{X\to 1} \frac{h(c)}{h(X)} = -A$$

$$1 = \frac{h(c)}{h(1)} = -A$$

$$A = -h(c)$$

AND THEREFORE (I) BECOMES:

$$-Y = \bar{p}^{o} + h(c)\ln\left|\frac{-h(c)+\bar{p}^{o}}{-h(c)+1}\right| - 1 = \bar{p}^{o} + h(c)\ln\left|\frac{h(c)-\bar{p}^{o}}{h(c)-1}\right| - 1$$

COMPOSITE SOLUTION

$$p^{*}(X) = p^{o}(X) + \bar{p}^{o}\left(\frac{X-1}{\varepsilon}\right) - (\text{COMMON PART})$$

$$= \frac{h(c)}{h(X)} + \bar{p}^{o}\left(\frac{X-1}{\varepsilon}\right) - h(c)$$