

Math 379 – Homework 10 (yay!)

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Friday, December 2, 2016

Problem 1: This problem refers to Example 4: Earth-Moon Spacecraft Problem.

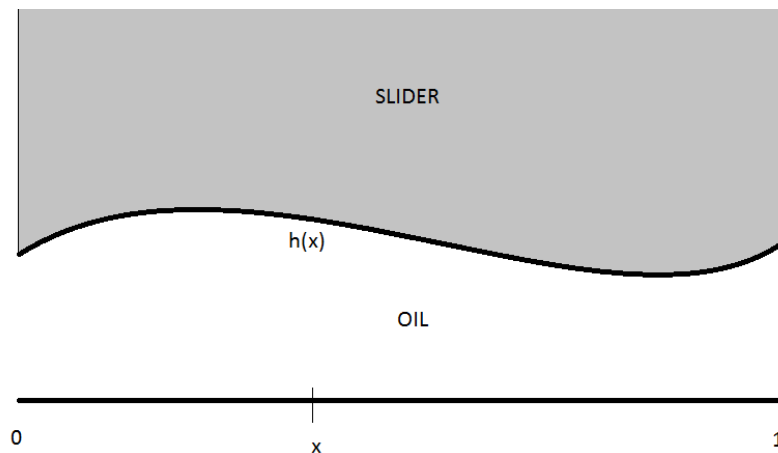
Show that the limiting velocity V_∞ satisfies

$$(V_\infty)^2 = \frac{\tan(\alpha)}{b}$$

Hint: Use the formula that relates V_∞ to V_0 , as well as the definition of e (at the beginning of the section on classical orbits), the formula for e in terms of α , and finally the definition of b .

Problem 2: [Reynold's equation for sliders]

Consider a slider that you put in oil, as in the following picture:



This slider exerts a certain pressure p^ϵ which satisfies the following ODE:

$$\begin{cases} -\epsilon ((h^3) p^\epsilon p_x^\epsilon)_x = (p^\epsilon h)_x \\ p^\epsilon(0) = p^\epsilon(1) = 1 \end{cases} \quad (\text{ODE})$$

Here $p^\epsilon = p^\epsilon(x)$, $x \in (0, 1)$, $h = h(x) : [0, 1] \rightarrow (0, \infty)$ is a given height, with $h(1) = 1$, and ϵ is a ‘viscosity coefficient’. We expect there to be a boundary layer near $x = 1$, and our goal is to find a good approximation p^* of p^ϵ .

(a) [Outer solution, near $x = 0$]

Apply the Ansatz:

$$p^\epsilon(x) = p^0(x) + \epsilon p^1(x) + \dots$$

Compare the $O(1)$ -terms and get an equation for $p^0(x)$. Impose $p^0(0) = 1$, solve for p^0 in terms of $h(0)$ and $h(x)$.

(b) [Inner solution, near $x = 1$]

Let $y = \frac{x-1}{\epsilon}$ and $\bar{p}_\epsilon(y) = p_\epsilon(x)$, $\bar{h}(y) = h(x)$.

Rewrite (ODE) in terms of \bar{p}_ϵ and \bar{h} , and apply the Ansatz:

$$\bar{p}_\epsilon(y) = \bar{p}_0(y) + \epsilon \bar{p}_1(y) + \dots,$$

and moreover Taylor expand the function $\bar{h}(y) = h(1 + \epsilon y)$.

Compare the $O(1)$ -terms and recall $h(1) = 1$, and obtain the ODE

$$\bar{p}_0 (\bar{p}_0)_y + \bar{p}_0 = -A$$

Solve for y in terms of \bar{p}_0 using separation of variables; at some point, it may be useful to note that

$$\frac{\bar{p}_0}{A + \bar{p}_0} = 1 - \frac{A}{A + \bar{p}_0}.$$

Impose the condition $\bar{p}_0(0) = 1$ (which is equivalent to $p^0(1) = 1$) and ultimately obtain that

$$-y = \bar{p}_0 - A \ln \left| \frac{A + \bar{p}_0}{A + 1} \right| - 1, \quad (\text{I})$$

where A is to be determined (note that there are many equivalent ways to write your answer) This gives us an implicit formula for $\bar{p}_0(y)$ in terms of A .

(c) [Matching]

Here, the matching condition is

$$\lim_{x \rightarrow 1} p^0(x) = \lim_{y \rightarrow -\infty} \bar{p}_0(y)$$

Since we want a finite answer in (I), the only way this works if the term in our \ln goes to 0^+ , and this is achieved only if \bar{p}_0 goes to $-A$. In other words, the right-hand-side of our matching condition has to be $-A$. Use this to solve for A and rewrite (I) in terms of the answer A that you found (this should be quick); this gives us an implicit formula for \bar{p}_0 . Finally, calculate your composite solution $p^*(x)$; leave the \bar{p}_0 term as $\bar{p}_0\left(\frac{x-1}{\epsilon}\right)$.