HW 1 – SOLUTIONS

PROBLEM 1

1) FROM (6) IN LECTURE, WE HAVE

$$p' = g'(e^\circ)e',$$

HENCE

$$\underline{p'_{tt}} = g'(e^\circ)e'_{tt}$$

$$= g'(e^\circ)\,(c_0)^2\Delta e'$$

$$= (c_0)^2\Delta\big(g'(e^\circ)e'\big)$$

$$= \underline{(c_0)^2\Delta p'}$$

2) START WITH (5)

$$e^\circ\,\underline{U}'_t = -\nabla p'$$

DIFF. WRT t

$$\underline{e^\circ\,\underline{U}'_{tt}} = -\nabla p'_t$$

$$= -\nabla\big(g'(e^\circ)e'_t\big) \qquad (\text{BY } (6))$$

$$= -g'(e^\circ)\nabla e'_t$$

$$= -g'(e^\circ)\,\nabla\big(-e^\circ\,\mathrm{DIV}(\underline{U}')\big) \quad (\text{BY } (4))$$

$$= e^\circ g'(e^\circ)\,\nabla\,\mathrm{DIV}(\underline{U}')$$

AND WE GET THE RESULT AFTER CANCELLING OUT $e^{\circ} \neq 0$.

PROBLEM 2

(a) START W/

$$-(U_\varepsilon)'' + V(x)\, U_\varepsilon + \varepsilon W U_\varepsilon = \lambda_\varepsilon U_\varepsilon$$

$$-(U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots)'' + V(x)(U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots)$$

$$+ \varepsilon W (U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots) = (\lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \ldots)(U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots)$$

$$= -U_0'' - \varepsilon U_1'' - \varepsilon^2 U_2'' + \ldots + V(x) U_0 + \varepsilon V(x) U_1 + \varepsilon^2 V(x) U_2 + \ldots$$

$$+ \varepsilon W U_0 + \varepsilon^2 W U_1 + \varepsilon^3 W U_2 = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \varepsilon^k \lambda_j U_{k-j} \quad \text{(BY HINT)}$$

O(1) - TERMS $\quad -U_0'' + V(x) U_0 = \lambda_0 U_0 \quad$ (THE ORIGINAL EQ !)

$O(\varepsilon^k)$ - TERMS $\quad -U_k'' + V(x) U_k + W U_{k-1} = \sum_{j=0}^{k} \lambda_j U_{k-j}$

PREVIOUS TERM

$$= \lambda_0 U_k + \sum_{j=1}^{k} \lambda_j U_{k-j}$$

REARRANGING, WE GET :

$$\boxed{-U_k'' + (V(x) - \lambda_0) U_k = -W U_{k-1} + \sum_{j=1}^{k} \lambda_j U_{k-j}}$$

(b) $$U_k = U_0 W_k$$

$$U_k' = U_0' W_k + U_0 W_k'$$

$$U_k'' = U_0'' W_k + U_0' W_k' + U_0' W_k' + U_0 W_k''$$

$$= U_0'' W_k + 2 U_0' W_k' + U_0 W_k''$$

HENCE (1) BECOMES:

$$-U_0'' W_k - 2 U_0' W_k' - U_0 W_k'' + (V - \lambda_0) U_0 W_k$$

$$= -W U_0 W_{k-1} + \sum_{j=1}^{k} \lambda_j U_0 W_{k-j}$$

$$= U_0 \left(-W W_{k-1} + \sum_{j=1}^{k} \lambda_j W_{k-j} \right)$$

MULTIPLY THIS BY U_0 TO GET:

$$-U_0'' W_k U_0 - 2 U_0' W_k' U_0 - U_0^2 W_k'' + (V - \lambda_0) U_0^2 W_k \tag{*}$$

$$= U_0^2 \left(-W W_{k-1} + \sum_{j=1}^{k} \lambda_j W_{k-j} \right)$$

BUT WE ALSO KNOW $-U_0'' + V U_0 = \lambda_0 U_0$, SO

MULTIPLY THIS BY $U_0 W_k$ TO GET

$$-U_0'' U_0 W_k + (V - \lambda_0) U_0^2 W_k = 0$$

THEREFORE (*) SIMPLIFIES TO:

$$-2\,U_0'\,W_k'\,U_0 - U_0^2\,W_k''$$

$$= (U_0)^2\left(-W\,w_{k-1} + \sum_{j=1}^{k} \lambda_j\, w_{k-j}\right)$$

BUT THIS IS JUST $-(U_0^2)'\,W_k' - (U_0)^2\,W_k''$

$$= -\left(U_0^2\,W_k'\right)'$$

SO $-\left(U_0^2\,W_k'\right)' = (U_0)^2\left(-W\,w_{k-1} + \sum_{j=1}^{k} \lambda_j\, w_{k-j}\right)$

$$\boxed{\left(U_0^2\,W_k'\right)' = (U_0)^2\left(W\,w_{k-1} - \sum_{j=1}^{k} \lambda_j\, w_{k-j}\right)}$$

?

(c) INTEGRATING THIS WITH [illegible] (NO TERMS [illegible] ASYMPTOTES!)

WE HAVE

$$0 = \int_{-\infty}^{\infty} (U_0)^2\left(W\,w_{k-1} - \sum_{j=1}^{k} \lambda_j\, w_{k-j}\right)$$

$$= \int_{-\infty}^{\infty} (U_0)^2\,W\,w_{k-1} - \underbrace{(U_0)^2\,\lambda_k\,\overset{1}{w_0}} - \sum_{j=1}^{k-1} (U_0)^2\,\lambda_j\, w_{k-j}$$

SO $\lambda_k \displaystyle\int_{-\infty}^{\infty} (U_0)^2 = \int_{-\infty}^{\infty} (U_0)^2\left(W\,w_{k-1} - \sum_{j=1}^{k-1} \lambda_j\, w_{k-j}\right)$ (ISOLATE λ_k)

$$\lambda_k = \left(\int_{-\infty}^{\infty} (U_0)^2\left(W\,w_{k-1} - \sum_{j=1}^{k-1} \lambda_j\, w_{k-j}\right)\right) \Big/ \int_{-\infty}^{\infty} (U_0)^2$$

FINALLY, USE $(U_0)^2\,W_{k-1} = U_0\,(U_0\,W_{k-1}) = U_0\,U_{k-1}$, SAME FOR $(U_0)^2\,W_{k-j}$

(d) FINALLY INTEGRATING OVER $(-\infty, t)$, WE GET:

$$(U_0)^2(t)\; W_k'(t) = \int_{-\infty}^{t} (U_0)^2(s)\left[W(s)\, W_{k-1}(s) - \sum_{j=1}^{k} \lambda_j\, W_{k-j}(s) \right]$$

$$W_k'(t) = \frac{1}{(U_0)^2(t)} \int_{-\infty}^{t} U_0(s)\left[W(s)\, \underbrace{(U_0 W_{k-1})}_{U_{k-1}}(s) - \sum_{j=1}^{k} \lambda_j\, \underbrace{(U_0 W_{k-j})}_{U_{k-j}}(s) \right]$$

NOW INTEGRATING OVER $(-\infty, x)$, AND USING $W_k(-\infty) = 0$,
WE GET

$$W_k(x) = \int_{-\infty}^{x} \frac{1}{(U_0)^2(t)} \int_{-\infty}^{t} U_0(s)\left[W(s)\, U_{k-1}(s) - \sum_{j=1}^{k} \lambda_j\, U_{k-j}(s) \right]$$

AND MULTIPLYING BY $U_0(x)$, WE GET

$$\underbrace{W_k(x)\, U_0(x)}_{U_k(x)} = U_0(x) \int_{-\infty}^{x} \frac{1}{(U_0)^2(t)} \int_{-\infty}^{t} U_0(s)\left[W(s)\, U_{k-1}(s) - \sum_{j=1}^{k} \lambda_j\, U_{k-j}(s) \right]$$