Math 379 – Homework 1

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Problem 1: Show that P^1 and $\mathbf{u}^1$ in Example 1 from lecture (acoustic approximation in fluid mechanics) satisfy the following PDE, where $c_0 := \sqrt{g'(\rho^0)}$ (assume $\rho^0 \neq 0$):

$$\begin{cases} P^1_{tt} - c_0^2\,\Delta P^1 = 0 \\ \mathbf{u}^1_{tt} - c_0^2\,\nabla\left(\operatorname{div}\mathbf{u}^1\right) = 0. \end{cases}$$

Problem 2: [Harmonic Oscillator]

Let $u_0 = u_0(x)$ be a solution of

$$-u_0'' + V(x)u_0 = \lambda_0 u_0 \tag{H}$$

subject to $\lim_{|x|\to\infty} u_0 = 0$, where $x \in \mathbb{R}$, $\lambda_0 \in \mathbb{R}$, and V is a given function.

And suppose we want to solve the following equation as a perturbation of (H),

$$-u_\epsilon'' + V(x)u_\epsilon + \epsilon W u_\epsilon = \lambda_\epsilon u_\epsilon \tag{H_ϵ}$$

subject to $\lim_{|x|\to\infty} u_\epsilon = 0$, where $W = W(x)$ is a given function.

(a) Expand u_ϵ and λ_ϵ out as

$$u_\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$
$$\lambda_\epsilon = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots$$

Plug this expansion into (H_ϵ) and show that the $O(\epsilon^k)$-terms (for $k = 1, 2, \cdots$) give you the equation

$$-u_k'' + (V - \lambda_0)u_k = -Wu_{k-1} + \sum_{j=1}^{k} \lambda_j u_{k-j}. \tag{1}$$

What do you get when you compare the $O(1)-$terms?

Hint: The following formula about the product of two series might be useful:

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j}$$

(b) Let's look for solutions of the form

$$u_k(x) = u_0(x)w_k(x),$$

where w_k is to be found.

Plug this into (1), and multiply by u_0 and obtain

$$\left(u_0^2 w_k'\right)' = u_0^2\left[Ww_{k-1} - \sum_{j=1}^{k} \lambda_j w_{k-j}\right] \tag{2}$$

Note: u_0 plays the role an integrating factor here, it's sort of like multiplying by $e^{\int P}$ in calculus, when you're trying to solve for $y''+Py'+Qy = 0$.

(c) Integrate (2) over $\mathbb{R}$, assuming that

$$u_0^2 w_k' \to 0 \qquad \text{as } |x| \to \infty$$

and solve for λ_k to obtain

$$\lambda_k = \frac{\int_{-\infty}^{\infty} u_0 \left[W u_{k-1} - \sum_{j=1}^{k-1} \lambda_j u_{k-j} \right]}{\int_{-\infty}^{\infty} u_0^2}.$$

This gives us a recursive definition of λ_k

(d) Integrate (2) over $(-\infty, t)$ for $t > 0$ (again assuming that $u_0^2 w_k'$ goes to 0 at $-\infty$), then integrate over $(-\infty, x)$ (assuming w_k goes to 0 at $-\infty$), and finally use $u_k = u_0 w_k$ to conclude

$$u_k(x) = u_0(x) \int_{-\infty}^{x} \frac{1}{u_0^2(t)} \int_{-\infty}^{t} u_0(s) \left[W(s) u_{k-1}(s) - \sum_{j=1}^{k} \lambda_j u_{k-j}(s) \right] ds dt.$$

This gives us a recursive definition of u_k.

Note: This problem is an example of Rayleigh-Schrödinger perturbation theory.