MATH 379 – FINAL EXAM – STUDY GUIDE

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The Math 379 – Final Exam is a 6 hour take-home exam that will take place (at your convenience) anytime between **Saturday, December 10** and **Sunday, December 18**, inclusive. It covers everything that we've learned, up to and including Example 8: "The Crushed Ice Problem." It will be a closed book and closed notes-exam, and no cheat-sheets will be allowed, so think of it like a 6h final exam, except with the flexibility of taking it whenever and wherever you want. Just for the midterm, you are not expected to memorize any formulas, unless I explicitly tell you otherwise below.

This is a study guide for the final, and (hopefully) covers everything you need to know for the exam. As before, make sure to study everything that's been done in lecture and on the homework. For your convenience, I have also combined this with the midterm study-guide.

Main concepts: Asymptotic Expansions (Chapters 1 and 3), Laplace's Method (Chapter 2), Boundary Layers (Chapter 4)

1. Chapter 1: Introduction

1.1. Example 1: Acoustic Approximation in Fluid Mechanics.

- Make sure you are comfortable with all the notation I have introduced, like $Du, \nabla u, \Delta u, D^2 u$, as well as the notation for vector-values functions, like $D\mathbf{u}$, $div(\mathbf{u})$. In case you're still confused about it, refer to the notation-handout I had in class.
- This is a complicated example, but hopefully it's easier now that you're more comfortable with multiple scales. The most important thing I'd like you to know how to do is to do an Ansatz and plug it into an equation, and compare term by term. You are not expected to figure out exactly what equations our ρ^1 satisfies, but I *could* ask you: Show that ρ^1 solves equation X.
- Know the notation $f = o(\epsilon)$ and $f = O(\epsilon)$ and its variants like $f = o(\epsilon^2)$ etc.

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- Know the fact that says that if $a_0 + a_1\epsilon + \cdots = b_0 + b_1\epsilon + \cdots$, then for all i, $a_i = b_i$ and know how to prove it.

1.2. Example 2: Perturbation of eigenvalues.

- You can certainly ignore this section, because it involves a bunch of linear algebra you're not responsible for. But again, know at least how to do the Ansatz and what you get when you compare the $O(1)$ and $O(\epsilon)$ terms.

1.3. Example 3: Derivation of the KdV equation.

- As mentioned many many times, this is a very ridiculous problem, and you really don't need to understand the details of it (especially since there's this one very weird trick at the end)
- However, I would say you do need to know how to simplify your equations after you do the change of variables (see Problem 1 on HW 2), how to do the Ansatz, and figure out what to conclude what equations you get **at least** for (A), (B), and (C), but **don't** worry what happens afterwards for (D) and how to combine them to get a KdV equation
- As far as the section on the theory of the KdV equation goes, you do **not** need to know what the general KdV equation looks like, and you do **not** need to know what the form of the solutions looks like. **BUT** once I tell you what they look like, you **DO** need to know how to plug them in, andyou **DO** need to know that trick of multiplying your equation by φ'.

1.4. Theoretical Aspects.

- **Know the definition of an asymptotic expansion, that is know what $f(\epsilon) \sim \sum_{k=0}^{\infty} a_k \epsilon^k$ means**
- Know how to prove the lemma about uniqueness of asymptotic expansions
- Know how to show that $e^{-\frac{1}{\epsilon}}$ has asymptotic expansion 0
- In Lecture 6 (Wednesday, September 21), I have proved another lemma about constructing a function with a given asymptotic expansion. You don't need to memorize the proof, but make sure to understand at least the main steps. In particular, make sure to know what a support function is.
- Know the more general definition of asymptotic expansion i.e. $f \sim \sum_{k=0}^{\infty} a_k \phi_k(\epsilon)$.

2. Chapter 2: Asymptotic Evaluation of Integrals

Disclaimer: You do NOT need to know the formulas for Laplace's method and Stationary phase, I will provide them for you if necessary!

2.1. Laplace's Method.

- Know what the assumptions on a and φ of Laplace's method are.
- Know what $I[\epsilon]$ is
- You do **NOT** need to know the (crazy) proof of Laplace's method in the special case, you can skip it if you like.
- Know the statement of Morse Lemma and why we need it
- You do not need to memorize the proof of the general Laplace's method, but at least know the general ideas.
- That said, at least know the outline behind what we're doing with Laplace's method, that is you start with the special case, and then use Morse Lemma to reduce the general case to the special case.
- Know how to calculate the first couple of terms in the general Laplace method, although I won't go too crazy :) In particular, know what η and ψ represent.
- Know the formula $\int_{\mathbb{R}} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$ (you can always reprove it in case you don't remember)
- Know what to do in case $\varphi(0) \neq 0$ or the maximum is at x_0 instead of 0
- Know what the multi-index notation is, and memorize Taylor's forula in several variables
- Again, know everything above, but in several dimensions

2.2. Stationary Phase.

- You can skip this section if you want to.

2.3. Application: Group vs. Phase Velocity.

- You can also skip this section if you want to

3. Chapter 3: Multiple Scales

3.1. Example 1: Rapidly Oscillating Coefficients.

- Know how to apply your usual Ansatz to this equation; you don't need to know why the Ansatz doesn't work
- Know how to apply your better Ansatz to this equation, and find the $O(\frac{1}{\epsilon^2})$, $O(\frac{1}{\epsilon})$ and $O(1)$ terms.
- Know the trick of multiplying by u^0 and then showing that $u^0_y = 0$
- For the $O(\frac{1}{\epsilon})$-term, know how to rewrite your equation in divergence form

- You **don't** need to know the trick of introducing the function w, but do know how to use it
- For the $O(1)$-term, know the trick of just integrating the equation and pulling out the terms that don't depend on y.

3.2. **Example 2: An oscillator with damping.**

- You don't need to know how to find the conserved quantity $g(t)$, but you do need to know how to show that it's conserved, and you do need to know how to deduce that u^ϵ is bounded from that.
- Again, know how to do the Ansatz
- Know how to solve the ODEs in the $O(\epsilon)-$term (like on your homework). You are responsible for undetermined coefficients, but not for variation of parameters.
- Know how to plug in the better Ansatz into your equation. Again, I will tell you which Ansatz to plug in, and know how to obtain the given equations for A and B.
- Know how we obtained the equations for A' and B'. You don't need to know how to write $\cos^3$ in terms of $\cos$ and $\sin$, but understand that we selected A and B to kill the resonance terms.
- You don't need to memorize Hamilton's equations; I will give them to you if necessary, but do know how to prove the lemma that $H(x(t), p(t))$ is a conserved quantity, and do know how to put an ODE in Hamiltonian form (all you need to do is to antidifferentiate H_x and H_p). Also, understand how to show that $|A|$ and $|B|$ are bounded from the fact that the Hamiltonian is constant.

3.3. **Example 3: WKB Method.**

- You don't need to understand why our first Ansatz didn't work, but you do need to know what to get when you plug in your Ansatz (don't worry about the part where I said skip the algebra)
- Know how to plug in the better Ansatz. I *will* give you the requirements on σ^ϵ, you don't need to figure them out, but you do need to figure out why we chose σ^ϵ to have the form that we want.
- Everything else in this example is fair game, especially how solve for A and B and how to figure out what u^ϵ looks like. You don't need to worry about the change-of-variables into θ-part

3.4. **Example 4: Nonlinear oscillator with damping.**

- I have to admit, this example (and the next example) are strange, and most of this is inappropriate to ask on an exam, but there are still a couple of things that you should still know how to do

- As usual, know how to plug in the guess into your ODE (I would give you that guess), and know how to plug in the Ansatz, and know how to show that the energy is constant (I would give you the energy)
- Don't worry about the weird change of variables thing to obtain $\omega^0(E)$.
- Understand why w solves a linear ODE, and know how to handle the $O(\epsilon)$ terms. I will tell you what to multiply your ODE with, but I won't tell you how to simplify the B-term
- You don't need to understand the fact that A is a function of E and a function of τ.

3.5. Example 5: Nonlinear wave equation.

- The same rules as Example 4 apply!

3.6. Example 6: A diffusion-transport PDE.

- Great news: Even though this example is fascinating, it's also inappropriate to ask on an exam, so you can beasically ignore most of it. The only thing you should know how to do is to plug in your Ansatz in your PDE and show that v^0 (which I'll provide) solves a linear PDE. Also know how to show that $v^0(\theta(t), t)$ is constant (given the ODE for θ, which I would provide).

3.7. Interlude: The Calculus of Variations.

- This, on the other hand, is a **very** important section (and my favorite topic) so make sure to thoroughly study it!
- Memorize the formula for the Euler-Lagrange equations, and **know** how to derive it in the $1-$dimensional case (which I did in lecture).
- Given a functional, derive its Euler-Lagrange equation
- Given a PDE, write it as an Euler-Lagrange equation for a certain functional $I[u]$ (try it for instane with $-\Delta u = f(u)$). This won't be always possible.
- Notice, by the way, that the minimal surface equation says that a minimal surface has 0 mean-curvature!

3.8. Example 7: An eikonal-continuity equation.

- Ignore the complex number business, if I asked you this question on the exam, I would directly give you the funtional $I^\epsilon[a^\epsilon, \theta^\epsilon]$
- Know how to find the $O(1)$-terms and know how to do the variation in both a^0 and θ^0 (see HW 8)

3.9. Example 8: Homogenization.

- Also an excellent example. Know how to do everything in this section, **except** I would tell you what $I[u^\epsilon]$ is and I would tell you the averaging trick.
- Don't worry about the multidimensional case

4. Chapter 4: Matched Asymptotic Expansions and Boundary Layers

4.1. Example 1: Introductory Example.

- This is a prototypical example and you should know how to do it in your sleep!
- For the exam, you're responsible of knowing how to do both methods (matching in asymptotic limits and matching in overlapping regions)
- Don't forget to show me the dominant balance, and in particular show why two of the three cases give you a contradiction
- In this example, you'd have to figure out what the matching condition is. I will tell you where the boundary layer is.

4.2. Example 2: Higher Orders.

- Of course this is a long example, but that doesn't mean that I won't ask you to do at least parts of it, so make sure to know how to do every step. In particular, know how to solve the inhomogeneous ODEs!
- Remember to find the inner solution recursively: First find $\overline{u}_0$, do matching to find the constant, and *then* find $\overline{u}_1$.
- On the exam, I would tell you whether to go to the $O(1)$ terms or to the $O(\epsilon)$ terms, so there won't be any confusion.
- Remember that in lecture, I made a mistake: You impose $u^1(1) = 0$, and $\bar{u}^1(0) = 0$, not 1.
- In this example, you'd have to figure out what the matching condition is. I will tell you where the boundary layer is.

4.3. Example 3: An internal layer.

- Another excellent example. Know how to solve all the ODEs in this example. Don't forget that u^0 is now piecewise defined.
- I would tell you what the matching condition is

4.4. Example 4: Earth-Moon Spacecraft Problem.

- Great news! (again) While this is a physical exciting example, this is also too inappropriate to ask for an exam, so you can ignore most of it. The only things I could ask you is how to solve for $x_0(t)$.

Of course I would provide you what $t^\star$ is. Everything else you can basically ignore

4.5. Example 5: A singular variational Problem.

- I think this is also a wonderful exam problem because it really has everything in it.
- Know how to derive the Euler-Lagrange equation and know how to find the outer-solution. I would tell you that you can ignore the $0-$solution
- Don't worry about the derivation that $\alpha = 1$ (that's embarrassingly nonrigorous)
- I would provide you with the change-of-variable, but you should know how to find the new PDE, and the $O(1)$ terms and the $O(\epsilon)$ terms
- Of course I would tell you the assumptions of s^ϵ at 0, and I would tell you to evaluate your terms at $(0, \dots, 0, y_n)$, and I would tell you that the ODE has a unique solution
- That said, you should know how to analyze the ODE for the $O(\epsilon)$terms and to derive that s^0 satisfies Laplace's equation at 0.
- **Given** the formula for mean-curvature in terms of s (which I would provide) know how to derive that this implies that our curve has mean curvature 0 at 0.

4.6. Example 6: A singular reaction-diffusion PDE.

- **Warning:** Just because I didn't go through it in detail doesn't mean that this is not an appropriate exam question! In particular, know how to derive all the claims that I made in this example. Of course, the same rules as Example 5 apply.
- Also **given** the formula for normal velocity, know how to derive that the normal velocity at 0 equals to the mean curvature at 0.

4.7. Example 7: Singular Perturbations of Eigenfunctions.

- Also a very standard and classical example. Know how to apply the boundary layer method to this example. Of course I would guide you a little bit since some of the steps are not obvious. For instance, I would give you the fact that $\bar{u}_0(y) = A + \frac{B}{|y|}$. I wouldn't tell you what the matching assumption is, but I would tell you how to obtain our "guess."
- As far as the Analysis of $O(\epsilon)$ goes, even though it's proofy, you should know how to do this, except the only thing I would tell you is that you have to multiply your $O(\epsilon)$ terms by u^0 and integrate on W_δ.

- In particular, know the integration by parts formula **with** boundary terms:

$$\int_W (\Delta u)v = \int_{\partial W} \frac{\partial u}{\partial \nu} v - \int_W Du \cdot Dv$$

- And know the definition of normal derivative $\frac{\partial u}{\partial \nu} = Du \cdot \nu$ and know the normal vector to $B(0, r)$ at $\mathbf{x}$, namely $\nu = \frac{\mathbf{x}}{r}$
- Last but not least, know that the derivative/gradient of $|\mathbf{x}|$ is $\frac{\mathbf{x}}{|\mathbf{x}|}$

4.8. Example 8: The Crushed Ice Problem.

- I won't ask you to know the proof of the derivation of the PDE for u_0, but do understand how to obtain the calculations in A and B.
- Know the formula

$$\int_W \Delta u = \int_{\partial W} \frac{\partial u}{\partial \nu},$$

but this is just a particular case of the above formula with $v = 1$.

5. Homework-Problems

The Homework Problems are a great practice for the exam question. If possible, try to do them without the hints if possible!

- **Homework 1:** Don't worry about Problem 1, but in Problem 2, do know how to do (a) (without the hint!), (b) (again, without any help, except that I will give you what u_k is), and (c) and (d) without the hint.
- **Homework 2:** Don't worry about Problem 1 (but know how to do it), **definitely** know how to do Problem 2 without the hints! (except in (c) I would tell you what the equation is)
- **Homework 3:** Problem 1 is also an excellent problem. In that problem, I would give you the equation and the form of u, but I wouldn't give you any hints, except I would tell you the substitution. Problem 2 is also a great Laplace method question. A good exam problem would be to do (b)-(d) without the hints, except that I would give you the statement of Laplace's method, as well as the formula in (a). Know the definition of $f(n) \sim g(n)$ as $n \to \infty$.
- **Homework 4:** Obviously I won't give you anything as ridiculous as Problem 1, except that I could ask you how to do it in a really special case, and I could ask you about how to calculate $L_0 a$ (but I'd give you all the formulas that you need, except I won't define η or ψ). And I would provide you with formulas for C_2 if necessary.

Problem 2 is also a great exam problem. Try to do it without the hint of second-order Taylor expansion and without the hint of the integral of $e^{-\frac{x^2}{2}}$.

- **Homework 5:** Also good problems, although Problem 2 is a bit inappropriate for the exam (this is not a calculus-course), but know how to do undetermined coefficients and how to solve second-order constant coefficient differential equations.
- **Homework 6:** Both problems are **excellent** exam problems. In Problem 1, know how to do the Ansatz and know why we chose A and B to kill the resonance terms, and know how to find the exact solution of the differential equation. For Problem 2, know how to plug in the Ansatz, know the trick of Taylor-expanding out $\sin(\theta^0 + \epsilon\theta^1)$ (I might not tell you about it!) Also know the trick of multiplying by a function and integrating by parts, and know how to do part (b). You don't need to know the formula for the integral of $\cos^2$.
- **Homework 7:** Ignore Problem 1 (it's too inappropriate), but Problem 2 is an excellent exam question (I would give you all the hints, except maybe I wouldn't tell you that you should integrate by parts)
- **Homework 8:** All the problems are **excellent** exam problems (including Problem 1c, for which I'd give you the hint). Know the integration-by-parts formula with 0-boundary terms (the HW had a typo):

$$\int_W Df \cdot Dg d\mathbf{x} = -\int_W div(Df) g d\mathbf{x}$$

- **Homework 9:** Again, great exam problems (especially Problem 1), but the same remarks above from Examples 1 and 2 from Chapter 4 apply.
- **Homework 10:** Ignore Problem 1, but definitely look at Problem 2. Try to do it without the hints if possible (except I would tell you what y is and what $\overline{p}_\epsilon$ and $\overline{h}(y)$ are.
- **Homework 11:** Also two great problems. In Problem 1, I would tell you what $\bar{u}_\epsilon$ is, but everything else is fair game. Try to do all of this without the hints, except I would tell you what to multiply your $O(\epsilon)$ term with and I would tell you what to do with the $\sin(2\bar{u}_0)$ terms. For Problem 2, I would give you all the hints.

And this is all you need to know. Any additional topics that I'll cover in lecture will not be on the exam. Good luck!