FINAL EXAM

SOLUTIONS

PROBLEM 1

(a) $f(\varepsilon) \sim \sum_{N=0}^{\infty} a_N \varepsilon^N$ MEANS $\forall K = 0, 1, \ldots$

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum_{N=0}^{K} a_N \varepsilon^N}{\varepsilon^K} = 0$$

(b) $f(\varepsilon) = e^{-\frac{1}{\varepsilon}}$ (FROM LECTURE)

(c) WE'LL DO THIS BY INDUCTION ON N

LET P_N BE THE PROPOSITION $a_i = b_i \quad \forall i = 0, 1, \ldots, N$

$N=0$ BY DEFINITION OF $\sim$ FOR $K=0$, WE GET

$$|a_0 - b_0| \leq |f(\varepsilon) - a_0| + |f(\varepsilon) - b_0|$$

$$= \frac{|f(\varepsilon) - a_0 \varepsilon^0|}{\varepsilon^0} + \frac{|f(\varepsilon) - b_0 \varepsilon^0|}{\varepsilon^0}$$

$$\longrightarrow 0 + 0 = 0$$

HENCE $a_0 = b_0$

SUPPOSE P_{N-1} IS TRUE, THAT IS $a_i = b_i \quad \forall i = 0, \ldots, N-1$

SHOW P_N IS TRUE, THAT IS $a_i = b_i \quad \forall i = 0, \ldots, N$

SO WE JUST NEED TO SHOW $a_N = b_N$

BY DEF OF $\sim$, WITH $K = N$, WE GET:

$$|a_N - b_N| = \frac{1}{\varepsilon^N}\left|\varepsilon^N a_N - \varepsilon^N b_N\right| = \frac{1}{\varepsilon^N}\left|\varepsilon^N a_N + \sum_{i=0}^{N-1} a_i\varepsilon^i - \sum_{i=0}^{N-1} a_i\varepsilon^i - \varepsilon^N b^N\right|$$

BY THE INDUCTIVE HYPOTHESIS

$$= \frac{1}{\varepsilon^N}\left|\sum_{i=0}^{N} a_N\varepsilon^N - \sum_{i=0}^{N} b_N\varepsilon^N\right|$$

$$\leq \underbrace{\frac{1}{\varepsilon^N}\left|\sum_{i=0}^{N} a_N\varepsilon^N - f(\varepsilon)\right|}_{\to 0} + \underbrace{\frac{1}{\varepsilon^N}\left|\sum_{i=0}^{N} b_N\varepsilon^N - f(\varepsilon)\right|}_{\to 0} \quad \text{(BY DEF OF } \sim\text{)}$$

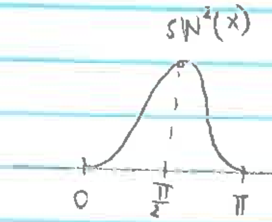
$\to 0$

BUT THIS ONLY [illegible] IF $a_N = b_N$

HENCE P_N IS TRUE

SO BY INDUCTION P_N IS TRUE $\forall N$, THAT IS $a_N = b_N$ $\forall N$

PROBLEM 2



LET $a(x) = x^2$ AND $\varphi(x) = 3 + \sin^2(x)$

NOTICE THAT φ HAS A MAX AT $\frac{\pi}{2}$ AND THAT MAX IS $= 4$

MOREOVER, NOTICE THAT

$$I[\varepsilon] = \int_0^{\pi} x^2 e^{\frac{\varphi(x)}{\varepsilon}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(U+\frac{\pi}{2}\right)^2 e^{\frac{\varphi(U+\frac{\pi}{2})}{\varepsilon}} dU \quad \left(U = x - \frac{\pi}{2}\right)$$

$$= \int_{-\pi/2}^{\pi/2} \left(x+\frac{\pi}{2}\right)^2 e^{\frac{\varphi(x+\frac{\pi}{2})}{\varepsilon}} dx$$

$$= e^{\frac{4}{\varepsilon}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\left(x+\frac{\pi}{2}\right)^2}_{\tilde{a}(x)} \underbrace{e^{\frac{\varphi(x+\frac{\pi}{2})-4}{\varepsilon}}}_{e^{\frac{\tilde{\varphi}(x)}{\varepsilon}}} dx$$

NOW THE FUNCTION $\tilde{\varphi}(x) = \varphi\left(x+\frac{\pi}{2}\right) - 4$ HAS A MAXIMUM OF 0 AT 0

AND HENCE

$$I[\varepsilon] \sim e^{\frac{4}{\varepsilon}} \left(\sqrt{\frac{2\pi\varepsilon}{|\tilde{\varphi}''(0)|}}\, \tilde{a}(0) + o(\sqrt{\varepsilon}) \right)$$

NOW $\tilde{\varphi}''(0) = \varphi''\left(\frac{\pi}{2}\right) = -2$ $\qquad \varphi'(x) = 2\sin(x)\cos(x) = \sin(2x)$

$\tilde{a}(0) = \left(0+\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$ $\qquad \varphi''(x) = 2\cos(2x)$

$\varphi''\left(\frac{\pi}{2}\right) = 2\cos(\pi) = -2$

SO $I[\varepsilon] \sim e^{\frac{4}{\varepsilon}} \left(\sqrt{\frac{\cancel{2}\pi\varepsilon}{\cancel{2}}}\, \frac{\pi^2}{4} + o(\sqrt{\varepsilon}) \right) = e^{\frac{4}{\varepsilon}} \sqrt{\pi\varepsilon}\, \frac{\pi^2}{4} + e^{\frac{4}{\varepsilon}} o(\sqrt{\varepsilon})$

PROBLEM 3

(a) MORSE LEMMA THERE ARE OPEN NEIGHBORHOODS U AND V OF 0 AND A SMOOTH DIFFEOMORPHISM $\psi : U \to V$ SUCH THAT

↳ (BIJECTION W/ ψ AND ψ^{-1} SMOOTH)

$$\varphi(\psi(y)) = -\frac{y^2}{2}$$

(b) WE NEEDED THAT BECAUSE WE FIRST PROVED LAPLACE'S METHOD IN THE CASE $\varphi(x) = -\frac{x^2}{2}$ AND USING MORSE, WE CAN REDUCE THE CASE OF GENERAL φ TO THE CASE $-\frac{x^2}{2}$.

PROBLEM 4

STEP 1 $\quad U^\varepsilon(t) = U^0(\sigma^\varepsilon(t), \varepsilon t) + \varepsilon U^1(\sigma^\varepsilon(t), \varepsilon t) + \dots$

$$(U^K)' = \frac{dU^K}{dt} = U^K_S(\sigma, \varepsilon t)\,\sigma' + \varepsilon U^K_T$$

$$(U^K)'' = U^K_{SS}(\sigma')^2 + 2U^K_{ST}\,\varepsilon\sigma' + \varepsilon^2 U^K_{TT} + U^K_S\,\sigma''$$

HENCE OUR ODE BECOMES:

$$U^0_{SS}\overset{O(1)}{(\sigma')^2} + 2\varepsilon U^0_{ST}\overset{O(1)}{\sigma'} + \varepsilon^2 U^0_{TT} + U^0_S\overset{O(\varepsilon)}{\sigma''}$$

$$+\varepsilon U^1_{SS}\overset{O(1)}{(\sigma')^2} + 2\varepsilon^2 U^1_{ST}\overset{O(1)}{\sigma'} + \varepsilon^3 U^1_{TT} + \varepsilon U^1_S\overset{O(\varepsilon)}{\sigma''}$$

$$+ e^{2\varepsilon t} U_0 + \varepsilon e^{2\varepsilon t} U_1 = 0$$

STEP 2 $\quad O(1)$ TERMS

$$(\sigma')^2 U^0_{SS} + e^{2\varepsilon t} U_0 = 0$$

CHOOSE σ SUCH THAT $(\sigma')^2 = e^{2\varepsilon t} \Rightarrow \sigma'(t) = e^{\varepsilon t}$

$$\Rightarrow \sigma(t) = \frac{1}{\varepsilon}e^{\varepsilon t} + C$$

REALITY CHECK

1) $\sigma(0) = \frac{1}{\varepsilon} + C = \frac{1}{\varepsilon} \Rightarrow C = -\frac{1}{\varepsilon} \Rightarrow \sigma(t) = \frac{1}{\varepsilon}e^{\varepsilon t}$)

2) $\sigma'(t) = e^{\varepsilon t} = 1 + (\varepsilon t) + \dots$ (Taylor) $= O(1)$ ✓

3) $\sigma''(t) = \varepsilon e^{\varepsilon t} = \varepsilon + \varepsilon^2 t + \dots = O(\varepsilon)$ ✓

THEN $O(1)$ BECOMES $\cancel{e^{2\varepsilon t}}\, U^0_{ss} + \cancel{e^{2\varepsilon t}}\, U_0 = 0$

$$U^0_{ss} + U^0 = 0$$

$$\Rightarrow \quad U^0 = A(T)\cos(s) + B(T)\sin(s)$$
$$= A(\varepsilon t)\cos\left(\tfrac{1}{\varepsilon}e^{\varepsilon t}\right) + B(\varepsilon t)\sin\left(\tfrac{1}{\varepsilon}e^{\varepsilon t}\right)$$

STEP 3 $O(\varepsilon)$: $2U^0_{sT}(\sigma') + U^0_s \dfrac{\sigma''}{\varepsilon} + U^1_{ss}(\sigma')^2 + e^{2\varepsilon t} U_1 = 0$

$$e^{2\varepsilon t} U^1_{ss} + e^{2\varepsilon t} U^1 = -2e^{\varepsilon t} U^0_{sT} - U^0_s e^{\varepsilon t}$$

$$e^{2\varepsilon t}\left[U^1_{ss} + U^1\right] = e^{\varepsilon t}\Big(-2\left[A(T)\cos(s) + B(T)\sin(s)\right]_{sT}$$
$$- \left[A(T)\cos(s) + B(T)\sin(s)\right]_s\Big)$$

$$= e^{\varepsilon t}\Big(2A'(T)\sin(s) - 2B'(T)\cos(s) + A(T)\sin(s)$$
$$- B(T)\cos(s)\Big)$$

$$e^{\varepsilon t}\left[U^1_{ss} + U^1\right] = \big(-2B'(T) - B(T)\big)\cos(s)$$
$$+\big(2A'(T) + A(T)\big)\sin(s)$$

STEP 4 KILL THE RESONANCE TERMS, IE REQUIRE

$$\begin{cases} -2B'(T) - B(T) = 0 \\ 2A'(T) + A(T) = 0 \end{cases} \Rightarrow \begin{cases} B'(T) = -\frac{1}{2}B(T) \\ A'(T) = -\frac{1}{2}A(T) \end{cases}$$

$$\Rightarrow \quad B(T) = C_1 e^{-\frac{T}{2}}$$
$$A(T) = C_2 e^{-\frac{T}{2}}$$

CONCLUSION $U^0(t) = A(\varepsilon t)\cos(e^{\varepsilon t}) + B(\varepsilon t)\sin(e^{\varepsilon t})$

$$U^0(t) = C_1 e^{-\frac{\varepsilon t}{2}}\cos\left(\tfrac{1}{\varepsilon}e^{\varepsilon t}\right) + C_2 e^{-\frac{\varepsilon t}{2}}\sin\left(\tfrac{1}{\varepsilon}e^{\varepsilon t}\right)$$

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PROBLEM 5

(a) EULER-LAGRANGE PDE:

$$-\operatorname{DIV}\left(D_p L\left(DU(x), U(x), x\right)\right) + L_z\left(DU(x), U(x), x\right) = 0$$

(b) SUPPOSE U IS A MINIMIZER, AND LET V BE ARBITRARY,

LET $g(\tau) = I[U + \tau V]$

$$= \int_W L(U' + \tau V', U + \tau V, x)\, dx$$

THEN $$g'(\tau) = \int_W L_p(U' + \tau V', U + \tau V, x) V' + L_z(U' + \tau V', U + \tau V, x) V$$

$$= \int_W \left[-\left(L_p(U' + \tau V', U + \tau V, x)\right)' + L_z(U' + \tau V', U + \tau V, x)\right] V$$

SINCE U IS A MINIMIZER, WE HAVE

$$0 = g'(0) = \int_W \left[-\left(L_p(U', U, x)\right)' + L_z(U', U, x)\right] V\, dx = 0$$

SINCE V IS ARBITRARY, WE GET,

$$-\left(L_p(U', U, x)\right)' + L_z(U', U, x) = 0$$

(c) LET'S CALCULATE

$$g''(\tau) = \int_W L_{pp}(U'+\tau V', U+\tau V, x)\, V'V' + L_{pz}(U'+\tau V', U+\tau V, x)\, V'V + L_{zp}(U'+\tau V', U+\tau V, x)\, VV' + L_{zz}(U'+\tau V', U+\tau V, x)\, VV$$

$$= \int_W L_{pp}(U'+\tau V', U+\tau V, x)\,(V')^2 + 2L_{pz}(U'+\tau V', U+\tau V, x)\, VV' + L_{zz}(U'+\tau V', U+\tau V, x)\, V^2 \, dx$$

IN PARTICULAR, U IS A MINIMIZER IF AND ONLY IF $g''(0) \geq 0$

THAT IS:

$$\int_W L_{pp}(U', U, x)(V')^2 + 2L_{pz}(U', U, x)\, VV' + L_{zz}(U', U, x)\, V^2 dx \geq 0.$$

PROBLEM 6

STEP 1 OUTER SOLUTION (NEAR $x = \pi$)

STEP 1

ANSATZ $\quad U^{\varepsilon}(x) = U^{0}(x) + \varepsilon U^{1}(x) + \dots$

$$\varepsilon U^{0}_{xx} + \varepsilon^2 U^{1}_{xx} + U^{0}_{x} + \varepsilon U^{1}_{x} = \cos(x)$$

$O(1)$ $\quad U^{0}_{x} = \cos(x)$

$$\Rightarrow U^{0}(x) = \sin(x) + C$$

IMPOSE $\quad U^{0}(\pi) = \beta^3 \Rightarrow \sin(\pi) + C = \beta^3$

$$\Rightarrow \underline{C = \beta^3}$$

$$\boxed{U^{0}(x) = \sin(x) + \beta^3}$$

PART 2 INNER SOLUTION (NEAR $x = 0$)

STEP 2

LET $\quad Y = \dfrac{x}{\varepsilon^{\alpha}}, \quad \overline{U}^{\varepsilon}(Y) = U^{\varepsilon}(x)$

$$U^{\varepsilon}_{x} = \overline{U}^{\varepsilon}_{Y}\left(\frac{1}{\varepsilon^{\alpha}}\right), \quad U^{\varepsilon}_{xx} = \overline{U}^{\varepsilon}_{YY}\left(\frac{1}{\varepsilon^{2\alpha}}\right)$$

SO OUR ODE BECOMES

$$\varepsilon\left(\frac{1}{\varepsilon^{2\alpha}}\right)\overline{U}^{\varepsilon}_{YY} + \frac{1}{\varepsilon^{\alpha}}\overline{U}^{\varepsilon}_{Y} = \cos(\varepsilon^{\alpha} Y)$$

$$\underbrace{\varepsilon^{1-2\alpha}}_{A}\overline{U}^{\varepsilon}_{YY} + \underbrace{\varepsilon^{-\alpha}}_{B}\overline{U}^{\varepsilon}_{Y} = \underbrace{\varepsilon^{0}}_{C}\cos(\varepsilon^{\alpha} Y)$$

STEP 3 DOMINANT BALANCE

CASE 1 A $\sim$ B, C SMALLER

$$1-2\alpha = -\alpha \Rightarrow \alpha = 1$$

A $\sim$ B $\sim \varepsilon^{-1}$, C ~ 1, SMALLER ✓ BINGO

CASE 2 A $\sim$ C, B SMALLER

$$1-2\alpha = 0 \Rightarrow \alpha = \tfrac{1}{2}$$

A $\sim$ C ~ 1, B $\sim \varepsilon^{-\frac{1}{2}}$ NOT SMALLER $\Rightarrow\Leftarrow$

CASE 3 B $\sim$ C, A SMALLER

$-\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow y = x$ BUT THIS GIVES US THE OUTER SOLUTION.

CONCLUSION $\alpha = 1$, $y = \frac{x}{\varepsilon}$, AND OUR ODE IS:

$$\varepsilon^{-1}\,\overline{U}^{\varepsilon}_{yy} + \varepsilon^{-1}\,\overline{U}^{\varepsilon}_{y} = \cos(\varepsilon^{\alpha} y)$$

$$\overline{U}^{\varepsilon}_{yy} + \overline{U}^{\varepsilon}_{y} = \varepsilon \cos(\varepsilon^{\alpha} y)$$

STEP 4 ANSATZ $\overline{U}^{\varepsilon}(y) = \overline{U}^{0}(y) + \varepsilon\,\overline{U}^{1}(y) + \cdots$

$$\overline{U}^{0}_{yy} + \cdots + \overline{U}^{0}_{y} + \cdots = \varepsilon \cos(\varepsilon^{\alpha} y)$$

O(1) $\overline{U}^{0}_{yy} + \overline{U}^{0}_{y} = 0$ AUX $r^2 + r = 0 \Rightarrow r = 0, -1$

$$\overline{U}^{0}(y) = A + Be^{-y}$$

IMPOSE $\overline{U}^{\circ}(0) = \alpha$

$$A + B = \alpha \implies B = \alpha - A$$

$$\overline{U}^{\circ}(y) = A + (\alpha - A)e^{-Y}$$

PART 3 MATCHING

STEP 5 METHOD 1 MATCHING IN COMMON LIMIT

REQUIRE $\lim_{x \to 0} U^{\circ}(x) = \lim_{Y \to \infty} \overline{U}^{\circ}(Y)$

$$\lim_{x \to 0} sw(x) + \beta = \lim_{Y \to \infty} A + (\alpha - A)e^{-Y}$$

$$\beta = A$$

$A = \beta$, HENCE $\overline{U}^{\circ}(Y) = \beta + (\alpha - \beta)e^{-Y}$

METHOD 2 MATCHING IN OVERLAPPING REGION

SUPPOSE OVERLAPPING REGION IS $(\pi \varepsilon^{\alpha_2}, \pi \varepsilon^{\alpha_1})$ $\quad 0 < \alpha_2 < \alpha_1 < 1$

AND LET $z = \dfrac{x}{\varepsilon^{\beta}}$ BE AN INTERMEDIATE VARIABLE

SINCE WE'RE IN THE OVERLAPPING REGION, WE HAVE

$$U^{\circ}(x) = sw(x) + \beta = sw(\varepsilon^{\beta} z) + \alpha$$

$$\overline{U}^{\circ}(y) = A + (\alpha - A)e^{-Y} = A + (\alpha - A)e^{-\frac{x}{\varepsilon}}$$
$$= A + (\alpha - A)e^{-\frac{\varepsilon^{\beta} z}{\varepsilon}}$$
$$= A + (\alpha - A)e^{-\varepsilon^{\beta - 1} z}$$

RECEIVE $\lim_{\varepsilon \to 0} U^0(x) = \lim_{\varepsilon \to 0} \overline{U^0}(Y)$

BUT IF $\pi\varepsilon^{\alpha_2} < x < \pi\varepsilon^{\alpha_1}$

$\pi\varepsilon^{\alpha_2} < \varepsilon^\beta z < \pi\varepsilon^{\alpha_1}$

SO $\varepsilon^\beta z \to 0$ BY SQUEEZE

AND $\pi\varepsilon^{\alpha_2 - 1} < \varepsilon^{\beta-1} z < \pi\varepsilon^{\alpha_k - 1}$

SO $\varepsilon^{\beta-1} z \to \infty$ BY SQUEEZE

HENCE $\lim_{\varepsilon \to 0} U^0(x) = \lim_{\varepsilon \to 0} \overline{U^0}(Y)$ BECOMES

$$\lim_{\varepsilon \to 0} \sin(\underbrace{\varepsilon^\beta z}_{\to 0}) + \beta = \lim_{\varepsilon \to 0} A + (\alpha - A) e^{-\overbrace{\varepsilon^{\beta-1} z}^{\infty}}$$

$$\beta = A \implies A = \beta$$

HENCE $\overline{U^0}(Y) = \beta + (\alpha - \beta) e^{-Y}$

STEP 6 CONCLUSION

COMPOSITE SOLUTION:

$$U^*(x) = U^0(x) + \overline{U^0}(Y) - \overbrace{\text{COMMON PART}}^{\beta}$$

$$= \sin(x) + \cancel{\beta} + \beta + (\alpha - \beta) e^{-Y} - \cancel{\beta}$$

$$\boxed{U^*(x) = \sin(x) + \beta + (\alpha - \beta) e^{-\frac{x}{\varepsilon}}}$$

PROBLEM 7

PART 1) OUTER SOL, NEAR $(W(t))^{\pm}$

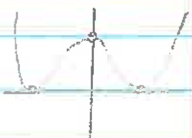
STEP 1 (a) ANSATZ $U^{\varepsilon}(x,t) = U^{0}(x,t) + \varepsilon U^{1}(x,t) + \cdots$

$$\varepsilon^{2}\left(U^{0}_{t} - \Delta U^{0}\right) + \Phi'(U^{0} + \varepsilon U^{1}) = \cdots$$

TAYLOR

$$\varepsilon^{2}\left(U^{0}_{t} - \Delta U^{0}\right) + \Phi'(U^{0}) + \varepsilon U^{1}\Phi''(U^{0}) = 0$$

$O(1)$ $\quad \Phi'(U^{0}) = 0 \Rightarrow U^{0}(x,t) \in \{-1, \cancel{0}, 1\},$



SINCE THE ONLY CRITICAL POINTS OF Φ ARE $-1, 0, 1$

IGNORE 0, AS NEVER ANSATZ $U^{0}(x,t) \in \{-1, 1\}$

PART 2) INNER SOLUTION, NEAR $\Gamma(t)$

(b) $\begin{cases} y_i = x_i & i = 1, \dots, N-1 \\ y_N = \dfrac{x_N - s^{\varepsilon}(x', t)}{\varepsilon} \end{cases}$

$$\overline{U}_{\varepsilon}(y,t) = U^{\varepsilon}\left(x', \frac{x_N - s^{\varepsilon}(x',t)}{\varepsilon}, t\right)$$

$i = 1, \dots, N-1$

$$U_{x_i} = \overline{U}_{y_i} + \overline{U}_{y_N}\left(-\frac{s_{x_i}}{\varepsilon}\right)$$

$$U_{x_i x_i} = \overline{U}_{y_i y_i} + \underbrace{\overline{U}_{y_i y_N}\left(-\frac{s_{x_i}}{\varepsilon}\right) + \overline{U}_{y_N y_i}\left(-\frac{s_{x_i}}{\varepsilon}\right)}_{2\overline{U}_{y_i y_N}\left(-\frac{s_{x_i}}{\varepsilon}\right)} + \overline{U}_{y_N y_N}\left(-\frac{s_{x_i}}{\varepsilon}\right)^{2} + \overline{U}_{y_N}\left(-\frac{s_{x_i x_i}}{\varepsilon}\right)$$

$i = N$

$$U_{x_N} = \overline{U}_{y_N}\left(\frac{1}{\varepsilon}\right)$$

$$U_{x_N x_N} = \overline{U}_{y_N y_N}\left(\frac{1}{\varepsilon^{2}}\right)$$

$i = t$

$$U_t = \overline{U}_{y_N}\left(-\frac{s_t}{\varepsilon}\right) + \overline{U}_t$$

So our PDE becomes:

$$\varepsilon^2\Big(\bar U_t - \Big(\frac{s_t}{\varepsilon}\Big)\bar U_{y_N} - \sum_{i=1}^{N-1}\Big[\bar U_{y_iy_i} + 2\bar U_{y_iy_N}\Big(-\frac{s_{x_i}}{\varepsilon}\Big) + \bar U_{y_Ny_N}\Big(\frac{-s_{x_i}}{\varepsilon}\Big)^2 + \bar U_{y_N}\Big(-\frac{s_{x_ix_i}}{\varepsilon}\Big)\Big] - \frac{1}{\varepsilon^2}\bar U_{y_Ny_N}\Big) + \Phi'(\bar U) = 0$$

(c) ANSÄTZE $\quad \bar U^\varepsilon(y,t) = \bar U^0(y,t) + \varepsilon \bar U^1(y,t) + \cdots$

$$s^\varepsilon(x',t) = s^0(x',t) + \varepsilon s^1(x',t) + \cdots$$

O(1) $\quad \Big(-\sum_{i=1}^{N-1} \bar U^0_{y_Ny_N}(s^0_{x_i})^2\Big) - \bar U^0_{y_Ny_N} + \Phi'(\bar U^0) = 0$

EVALUATE THIS AT $(0,0,\dots,0,y_N,0)$ AND USE $s^0_{x_i}(0,\dots,0,0)=0$ TO GET,

$$\underbrace{-\bar U^0_{y_Ny_N}(0,\dots,0,y_N,0)}_{} + \Phi'(\bar U^0(0,\dots,0,y_N,0)) = 0$$

$$-f'' + \Phi'(f) = 0$$

(d) O(ε)

$$-s^0_t\,\bar U^0_{y_N} - \sum_{i=1}^{N-1}\Big(-2\bar U^0_{y_iy_N}(\cancel{s^0_{x_i}}) + 2\bar U^0_{y_Ny_N}(\cancel{s^0_{x_i}})(s^1_{x_i}) + \bar U^1_{y_Ny_N}(\cancel{s^0_{x_i}})^2 - \bar U^0_{y_N}\,s^0_{x_ix_i}\Big)$$

$$-\bar U^1_{y_Ny_N} + \bar U^1\,\Phi''(\bar U^0) = 0$$

EVALUATE THIS AT $(0,\dots,0,y_N,0)$ AND USE $s^0_{x_i}(0,\dots,0)=0$

TO GET $\quad -s^0_t f' + f'\sum_{i=1}^{N-1} s^0_{x_ix_i} - h'' + h\,\Phi''(f) = 0$

$$-h'' + h\,\Phi''(f) - f'(s^0_t - \Delta s^0) = 0$$

(e) $$\int_{-\infty}^{\infty} \underbrace{-h'' f' + h\,\Phi''(f) f'}_{\text{IBP TWICE}} - \int_{-\infty}^{\infty} (f')^2 [s_t^0 - \Delta s^0] = 0$$

$$\int_{-\infty}^{\infty} -h f''' + h\,\Phi''(f) f' = [s_t^0 - \Delta s^0] \int_{-\infty}^{\infty} (f')^2$$

$$\int_{-\infty}^{\infty} h \underbrace{[-f''' + \Phi''(f) f']}_{0} = [s_t^0 - \Delta s^0] \int_{-\infty}^{\infty} (f')^2$$

BUT DIFF $-f'' + \Phi'(f) = 0$ TO GET $-f''' + \Phi''(f) f' = 0$

$$\Rightarrow [s_t^0 - \Delta s^0] \underbrace{\int_{-\infty}^{\infty} (f')^2}_{>0} = 0$$

$$\Rightarrow s_t^0 - \Delta s^0 = 0$$

$$\Rightarrow \boxed{s_t^0 = \Delta s^0} \quad \text{AT } (0, \ldots, 0)$$

PROBLEM 8

MULTIPLY THE FIRST EQUATION BY V AND INTEGRATE ON $B(0,r)$ TO GET:

$$\int_{B(0,r)} (\Delta U) V = \int_{B(0,r)} V^2$$

BUT BY INTEGRATION BY PARTS, WE GET

$$\int_{B(0,r)} V^2 = \int_{B(0,r)} (\Delta U) V = \int_{\partial B(0,r)} \left(\frac{\partial U}{\partial \nu}\right) V - \int_{B(0,r)} (DU)\cdot(DV)$$

$$= \int_{\partial B(0,r)} \left(\frac{\partial U}{\partial \nu}\right) V - \int_{\partial B(0,r)} U\left(\frac{\partial V}{\partial \nu}\right) + \int_{B(0,r)} U\underbrace{(\Delta V)}_{-U}$$

$$= \int_{\partial B(0,r)} \left(\frac{\partial U}{\partial \nu} V - U\frac{\partial V}{\partial \nu}\right) - \int_{B(0,r)} U^2$$

THEREFORE, WE GET

$$\int_{B(0,r)} U^2 + V^2 = \int_{\partial B(0,r)} \left(\frac{\partial U}{\partial \nu}\right) V - \int_{\partial B(0,r)} U\left(\frac{\partial V}{\partial \nu}\right)$$

$$A = B - C$$

NOW AS $r \to \infty$, $\int_{B(0,r)} U^2 + V^2 = \int_{\mathbb{R}^N} U^2 + V^2$

ON THE OTHER HAND,

$$\left|\text{B}\right| \le \int_{\partial B(0,r)} \left|\frac{\partial u}{\partial \nu}\right| |v(x)| \qquad \frac{\partial u}{\partial \nu} = Du \cdot \nu = Du \cdot \frac{x}{|x|}$$

$$\le \int_{\partial B(0,r)} \underbrace{|Du|}_{\le C} \underbrace{\left|\frac{x}{|x|}\right|}_{=1} \frac{C}{|x|^N}$$

$$= C \int_{\partial B(0,r)} \frac{C}{r^N}$$

$$= \frac{C}{r^N} \int_{\partial B(0,r)} 1$$

$$= \frac{C}{r^N} |\partial B(0,r)|$$

$$= C \frac{N\alpha(N) r^{N-1}}{r^N}$$

$$= \frac{C}{r} \longrightarrow 0 \text{ AS } r \to \infty$$

THEREFORE AS $r \to \infty$, $\text{B} \to 0$

SIMILARLY, AS $r \to \infty$, $\text{C} \to 0$

AND THEREFORE, AS $r \to \infty$, $\text{A} = \text{B} - \text{C}$ BECOMES

$$\int_{\mathbb{R}^N} \underbrace{u^2}_{\ge 0} + \underbrace{v^2}_{\ge 0} = 0$$, AND HENCE $u^2 = 0$ AND $v^2 = 0$

$u = 0$ AND $v = 0$ ON $\mathbb{R}^N$