MATH 379 – FINAL EXAM

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Name: ____

Instructions: Welcome to your final exam! You have 8 hours to take it, for a total of 100 points. Write in full sentences whenever you can. If you need to continue your work on the back of a page, please indicate that you're doing so, or else your work may be discarded. Good luck, and may the Ansatz be with you! :)

Honor Code: I agree to abide by the rules on the next page, and I promise to act with honesty and integrity.

Signature: ____

Note: After the exam, please try to send me a quick e-mail telling me that you've submitted the exam, so that I can pick it up and grade it.

1		10
2		10
3		5
4		15
5		15
6		15
7		20
8		10
Total		100

Date: Saturday, December 10, 2016 – Sunday, December 18, 2016.

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Rules:

(1) You are not allowed to exceed the allotted time limit.

(2) You have to take this exam in one day; that is, you are not allowed to take the exam for 4h, then sleep for 10h, and then continue the exam on the next day. But it's ok, for example, to work on it from 6 pm to 2 am (although I won't recommend it).

(3) No books/notes/cheat sheets are allowed, and you are not allowed to use personal electronic devices, including calculators. The only exception is when you want me to e-mail me with a question or a clarification.

(4) Meals or bathroom breaks do **not** count towards the 8h time limit, but try not to think of the exam during your break. Also, commuting to my office to ask me questions also does not count against the time limit.

(5) You are not allowed to communicate with anyone about this exam and its contents, *unless* that person is me, or both you and the person you're communicating with are already done with the exam.

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1. *(10 points)*

(a) *(2 points)* Define:

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n.$$

(b) *(2 points)* **Without proof**, give me an example of a nontrivial function $f(\epsilon)$ (i.e. not the 0 function) such that

$$f(\epsilon) \sim \sum_{n=0}^{\infty} 0 \, \epsilon^n$$

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(c) *(6 points)* Carefully show that asymptotic expansions are unique, in the sense that if

$$f(\epsilon) \sim \sum_{n=0}^{\infty} a_n \epsilon^n$$

and

$$f(\epsilon) \sim \sum_{n=0}^{\infty} b_n \epsilon^n$$

then $a_i = b_i$ for all $i \in \mathbb{N}$.

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2. *(10 points)* Laplace's method states that if φ is a smooth function that has a global max at 0 with $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi''(0) < 0$, and a is any smooth function (not necessarily with compact support), then, as $\epsilon \to 0$,

$$I[\epsilon] = \int_a^b a(x) e^{\frac{\varphi(x)}{\epsilon}} dx \sim \sqrt{\frac{2\pi\epsilon}{|\varphi''(0)|}} a(0) + o(\sqrt{\epsilon}).$$

Apply Laplace's method to find an asymptotic expansion of

$$\int_0^\pi x^2 \, e^{\frac{3+\sin^2(x)}{\epsilon}} \, dx$$

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3. *(5 points)*

(a) *(2 points)* State Morse's Lemma (in one dimension)

(b) *(3 points)* In roughly one or two sentences, explain why Morse's lemma is used to find an asymptotic expansion of

$$I[\epsilon] = \int_a^b a(x) e^{\frac{\varphi(x)}{\epsilon}} dx$$

In other words, what did we first do to evaluate this integral, and why did we need to use Morse's lemma?

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4. *(15 points)* Since you all liked the WKB method so much :)

   Consider the following ODE, where $u^\epsilon = u^\epsilon(t)$.

$$u_\epsilon'' + e^{2\epsilon t}\, u_\epsilon = 0$$

   Apply the following WKB-Ansatz:

$$u^\epsilon(t) = u^0(\sigma^\epsilon(t), \epsilon t) + \epsilon u^1(\sigma^\epsilon(t), \epsilon t) + \cdots$$

   where $u^k = u^k(s, \tau)$ and $\sigma^\epsilon = \sigma^\epsilon(t)$ is to be selected and use this to find an *explicit* formula for $u^0(t) = u^0(\sigma^\epsilon(t), \epsilon t)$.

   Requirements: (don't forget to check them)

$$\sigma^\epsilon(0) = \frac{1}{\epsilon}, \;\; \sigma_\epsilon'(t) = O(1), \;\; \sigma_\epsilon''(t) = O(\epsilon)$$

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5. *(15 points)* Suppose that u is a minimizer of

$$I[v] = \int_W L(Dv, v, x)dx$$

where W is an open subset of $\mathbb{R}^n$ and $L = L(p, z, x)$

(a) *(3 points)* What PDE must u satisfy? (just state the PDE)

(b) *(7 points)* In the case $n = 1$, show me how to derive the PDE in (a).

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(c) *(5 points)* Still in the case $n = 1$, notice that in theory maximizers of $I[u]$ also satisfy the same PDE! Show that u is a minimizer if and only if u satisfies the following additional condition called **convexity** of L: For all v, we have

$$\int_W L_{pp}(u', u, x)(v')^2 + 2L_{pz}(u', u, x)v'v + L_{zz}(u', u, x)L_{zz}v^2 dx \geq 0$$

Hint: Use the second derivative test from calculus!

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6. *(15 points)* Consider the following ODE on $(0, \pi)$, where $u^\epsilon = u^\epsilon(x)$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{cases} \epsilon u^\epsilon_{xx} + u^\epsilon_x = \cos(x) \\ u^\epsilon(0) = \alpha, u^\epsilon(\pi) = \beta \end{cases}$$

We expect there to be a boundary layer at $x = 0$. Find a good approximation $u^\star$ of u^ϵ that incorporates the fact that u^ϵ has a boundary layer. You only need to limit yourself to the terms of order $O(1)$. For the matching-part, you may use any method that you wish.

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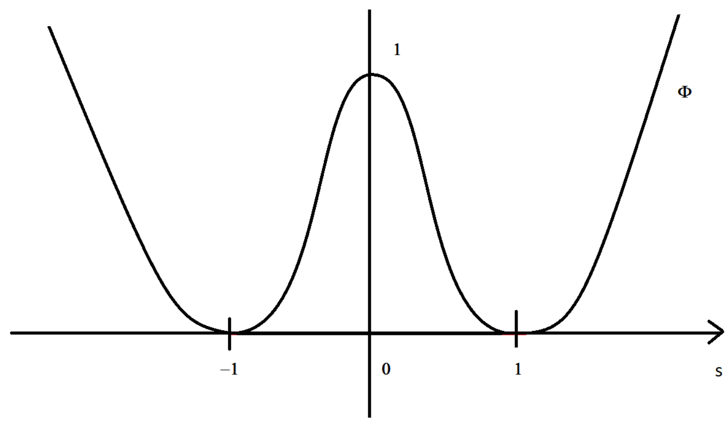
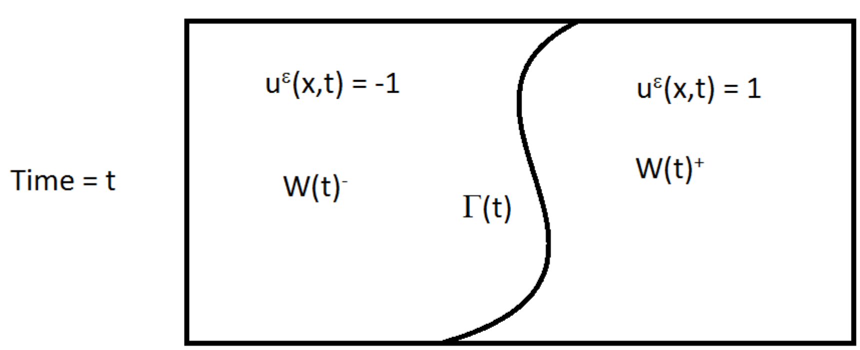
7. *(20 points)* Suppose $u^\epsilon = u^\epsilon(x,t)$ is a solution of the following reaction-diffusion PDE, where $g = g(x)$ is a fixed function.

$$\begin{cases} \epsilon^2(u^\epsilon_t - \Delta u^\epsilon) + \Phi'(u^\epsilon) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u^\epsilon(x,0) = g \end{cases}$$

Here $\Phi = \Phi(s)$ is a double-well potential function, that is a function with a local min at $s = \pm 1$ with $\Phi(\pm 1) = 0$, and a local max at $s = 0$ with $\Phi(0) = 1$, as in the following picture:



It turns out that for every t, there are two regions $W(t)^\pm$, where $u^\epsilon(x,t) \to \pm 1$ on $W(t)^\pm$, separated by a boundary layer $\Gamma(t)$, as in the following picture:



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(a) *(3 points)* [Outer Solution, near $W(t)^{\pm}$]

Apply the Ansatz:

$$u^{\epsilon}(x,t) = u^{0}(x,t) + \epsilon u^{1}(x,t) + \cdots$$

Find the $O(1)$ terms and conclude that $u^{0}(x,t) \in \{-1, 0, 1\}$.

Since 0 is not a minimizer, we can ignore it, and hence we get:

$$u^{0}(x,t) \in \{\pm 1\}$$

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(b) *(5 points)* [Inner Solution, near $\Gamma(t)$]

Suppose that the boundary layer is locally the graph of a function $x_n = s^\epsilon(x', t)$, where $x' = (x_1, \cdots, x_{n-1})$. Assume that $s^\epsilon(0,0) = 0$ and $s^\epsilon_{x_i}(0,0) = 0$ for $i = 1, \cdots, n-1$.

Define $y = (y_1, \cdots, y_n)$ by:

$$\begin{cases} y_i = x_i & (i = 1, \cdots, n-1) \\ y_n = \frac{x_n - s^\epsilon(x',t)}{\epsilon} \end{cases}$$

and let

$$\overline{u}_\epsilon(y,t) = u^\epsilon\left(x', \frac{x_n - s^\epsilon(x',t)}{\epsilon}, t\right)$$

Find a PDE that $\bar{u}_\epsilon$ satisfies (no need to simplify it)

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(c) *(3 points)* Apply the Ansätze:

$$\overline{u}_\epsilon(y,t) = \bar{u}_0(x,t) + \epsilon\overline{u}_1(x,t) + \cdots$$

and

$$s^\epsilon(x',t) = s^0(x',t) + \epsilon s^1(x',t) + \cdots$$

Find the $O(1)$ terms, and show that $f(y_n) = \overline{u}_0(0,\cdots,0,y_n,0)$ satisfies the ODE

$$-f'' + \Phi'(f) = 0.$$

(d) *(4 points)* Now find the $O(\epsilon)$ terms and show that $h(y_n) = \overline{u}_1(0,\cdots,0,y_n,0)$ satisfies the ODE

$$-h'' + \Phi''(f)h - f'(s^0_t(0,0) - \Delta s^0(0,0)) = 0$$

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(e) *(5 points)* Finally, multiply the equation in (d) by f' and integrate with respect to y_n on $\mathbb{R}$ (ignore the boundary terms) to get that, at $(0,0)$, we have

$$s_t^0(0,0) = \Delta s^0(0,0)$$

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8. *(10 points, harder)* Suppose that $u = u(x)$ and $v = v(x)$ are solutions to the following system of PDE in $\mathbb{R}^n$ (there is no typo here):

$$\begin{cases} \Delta u = v \\ -\Delta v = u \end{cases}$$

Assume that there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^n$ the following holds:

$$|u(x)| \leq \frac{C}{|x|^n}, |v(x)| \leq \frac{C}{|x|^n}, |Du(x)| \leq C, |Dv(x)| \leq C$$

Show that $u = v = 0$ in $\mathbb{R}^n$.

Hint: Fix $r > 0$, multiply the first equation by v and integrate on $B(0, r)$. You may use that the area of the sphere $\partial B(0, r)$ is $n\alpha(n)r^{n-1}$, where $\alpha(n)$ is the volume of the unit ball $B(0, 1)$ in $\mathbb{R}^n$ (you don't need to figure out $\alpha(n)$ to solve this problem)

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(Scratch work)

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(Scratch work)

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