PCMI 2017 - Introduction to Random Matrix Theory
Homework #11 – 07.13.2017

Exercise 1. (The Circular Beta Ensemble) Let $\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\}$, with $0 \leq \theta_k < 2\pi$ for any k, be the eigenvalues of the $n \times n$ circular beta ensemble. The probability density function of the phases θ_k is given by

$$f(\theta_1, \ldots, \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < j \leq n} |e^{i\theta_k} - e^{i\theta_j}|^\beta$$

Find the normalizing constants $Z_{n,1}$ and $Z_{n,2}$ for $n = 2$ and $n = 3$.

Exercise 2. Given an infinite sequence of coefficients $\alpha_0, \alpha_1, \ldots$ in $\mathbb{D}$ (the open unit disk in $\mathbb{C}$), we define 2×2 matrices

$$\Theta_k = \begin{pmatrix} \overline{\alpha_k} & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$. From these, we form block-diagonal matrices

$$\mathcal{L} = \operatorname{diag}\big(\Theta_0, \Theta_2, \Theta_4, \ldots\big) \quad \text{and} \quad \mathcal{M} = \operatorname{diag}\big(\Theta_{-1}, \Theta_1, \Theta_3, \ldots\big),$$

where $\Theta_{-1} = [1]$, the 1×1 identity matrix.

Prove that the matrix $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots) = \mathcal{L}\mathcal{M}$ is unitary. The matrix $\mathcal{C}$ is called the CMV matrix associated to the coefficients $\alpha_0, \alpha_1, \ldots$ (called the Verblunsky coefficients of $\mathcal{C}$).

Exercise 3. Suppose that the random variable $X_n : \Omega \to \mathbb{C}$ is uniformly distributed on the disk $D(0, \frac{C}{n^\gamma})$, with $C > 0$ and $\gamma > 0$. Find the variance of X_n.

Exercise 4. A random variable $X : \Omega \to \mathbb{C}$, with values in the unit disk $\mathbb{D}$, is Θ_ν-distributed (for $\nu > 1$) if

$$\mathbb{E}\left(f(X)\right) = \tfrac{\nu-1}{2\pi} \iint_{\mathbb{D}} f(z)(1 - |z|^2)^{(\nu-3)/2}\, d^2 z.$$

This means that the density of the random variable X is $\phi_\nu(z) = \frac{\nu-1}{2\pi}(1-|z|^2)^{(\nu-3)/2}\mathbf{1}_{\mathbb{D}}(z)$.

For $\nu \geq 2$ an integer, this has the following geometric interpretation: If v is chosen at random from the unit sphere S^ν in $\mathbb{R}^{\nu+1}$ according to the usual surface measure, then $v_1 + iv_2$ is Θ_ν-distributed.

Prove that if X is Θ_ν-distributed, then

$$\mathrm{Var}(X) = \tfrac{2}{\nu+1}.$$

Exercise 5. (Connection Between Self-adjoint Matrices and Unitary Matrices) Let $f : \mathbb{C} \setminus \{-i\} \to \mathbb{C}$ be defined by $f(z) = \frac{z-i}{z+i}$ (f is called the Cayley transform). Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a self-adjoint matrix. Prove that

$$f(A) = \frac{A - iI}{(A + iI)} = (A - iI)(A + iI)^{-1}$$

is a unitary matrix.

Let $g : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ be defined by $g(w) = i\frac{1+w}{1-w}$. Let $B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a unitary matrix such that 1 is not an eigenvalue for B. Prove that

$$g(B) = i\frac{I + B}{I - B} = i(I + B)(I - B)^{-1}$$

is a self-adjoint matrix.