Exercise 1. (The Circular Beta Ensemble) Let \( \{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\} \), with \( 0 \leq \theta_k < 2\pi \) for any \( k \), be the eigenvalues of the \( n \times n \) circular beta ensemble. The probability density function of the phases \( \theta_k \) is given by
\[
f(\theta_1, \ldots, \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < j \leq n} |e^{i\theta_k} - e^{i\theta_j}|^\beta
\]
Find the normalizing constants \( Z_{n,1} \) and \( Z_{n,2} \) for \( n = 2 \) and \( n = 3 \).

Exercise 2. Given an infinite sequence of coefficients \( \alpha_0, \alpha_1, \ldots \) in \( \mathbb{D} \) (the open unit disk in \( \mathbb{C} \)), we define \( 2 \times 2 \) matrices
\[
\Theta_k = \begin{pmatrix} \alpha_k & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix}
\]
where \( \rho_k = \sqrt{1 - |\alpha_k|^2} \). From these, we form block-diagonal matrices
\[
\mathcal{L} = \text{diag}(\Theta_0, \Theta_2, \Theta_4, \ldots) \quad \text{and} \quad \mathcal{M} = \text{diag}(\Theta_{-1}, \Theta_1, \Theta_3, \ldots),
\]
where \( \Theta_{-1} = [1] \), the \( 1 \times 1 \) identity matrix.

Prove that the matrix \( \mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots) = \mathcal{L} \mathcal{M} \) is unitary. The matrix \( \mathcal{C} \) is called the CMV matrix associated to the coefficients \( \alpha_0, \alpha_1, \ldots \) (called the Verblunsky coefficients of \( \mathcal{C} \)).

Exercise 3. Suppose that the random variable \( X_n : \Omega \to \mathbb{C} \) is uniformly distributed on the disk \( D(0, C \gamma) \), with \( C > 0 \) and \( \gamma > 0 \). Find the variance of \( X_n \).

Exercise 4. A random variable \( X : \Omega \to \mathbb{C} \), with values in the unit disk \( \mathbb{D} \), is \( \Theta_\nu \)-distributed (for \( \nu > 1 \)) if
\[
\mathbb{E}(f(X)) = \frac{\nu-1}{2\pi} \int_{\mathbb{D}} f(z)(1 - |z|^2)^{(\nu-3)/2} \, d^2z.
\]
This means that the density of the random variable \( X \) is \( \phi_\nu(z) = \frac{\nu-1}{2\pi} (1-|z|^2)^{(\nu-3)/2} \mathbb{1}_\mathbb{D}(z) \).

For \( \nu \geq 2 \) an integer, this has the following geometric interpretation: If \( \nu \) is chosen at random from the unit sphere \( S^\nu \) in \( \mathbb{R}^{\nu+1} \) according to the usual surface measure, then \( u_1 + iv_2 \) is \( \Theta_\nu \)-distributed.
Prove that if $X$ is $\Theta_\nu$-distributed, then

$$\text{Var}(X) = \frac{2}{\nu+1}.$$ 

**Exercise 5.** (Connection Between Self-adjoint Matrices and Unitary Matrices) Let $f : \mathbb{C} \setminus \{-i\} \to \mathbb{C}$ be defined by $f(z) = \frac{z+i}{z+i}$ ($f$ is called the Cayley transform). Let $A \in \text{Mat}_{n \times n}(\mathbb{C})$ be a self-adjoint matrix. Prove that

$$f(A) = \frac{A - iI}{(A + iI)} = (A - iI)(A + iI)^{-1}$$

is a unitary matrix.

Let $g : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ be defined by $g(w) = i\frac{1+w}{1-w}$. Let $B \in \text{Mat}_{n \times n}(\mathbb{C})$ be a unitary matrix such that 1 is not an eigenvalue for $B$. Prove that

$$g(B) = i\frac{I + B}{I - B} = i(I + B)(I - B)^{-1}$$

is a self-adjoint matrix.