PCMI 2017 - Introduction to Random Matrix Theory
Homework #9 – 07.10.2017

Exercise 1. Let $X_n = X_n(\omega)$ be an ensemble of $n \times n$ symmetric random matrices and let $\Lambda_n = \{\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_n(\omega)\}$ be the eigenvalues of X_n. Assume that there exists a nonnegative function $\sigma : \mathbb{R} \to \mathbb{R}$, supported on a compact interval $[a, b]$, continuous on $[a, b]$ and analytic on (a, b) and there exists a constant $c > 0$ such that

$$\frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{\lambda_k(\omega)}{n^c}} \xrightarrow{P} \sigma(x)\, dx$$

a) Let $x_0 \in (a, b)$ be such that $\sigma(x_0) > 0$. Find the correct scaling of the eigenvalues $b_n(\Lambda_n - a_n)$ near x_0.

b) Suppose that $\sigma(b) = 0$ and there exists a constant $d > 0$ such that $\sigma(b - z) = \alpha z^d + o(z^d)$ as $z \downarrow 0$ and $\alpha \neq 0$. Find the correct scaling of the eigenvalues $b_n(\Lambda_n - a_n)$ near b.

c) Apply the previous two parts to the Gaussian Beta Ensemble.

Exercise 2. The Laplace transform of a random variable X is by definition the function $L_X : \mathbb{R} \to \mathbb{R}$ defined by $L_X(t) = \mathbb{E}(e^{tX})$. Prove that if X is a Poisson random variable with parameter $\lambda > 0$, then its Laplace transform is the function $L_X(t) = e^{-\lambda(1-e^{-t})}$.

Exercise 3. Suppose calls are coming into a telephone exchange at an average rate of 3 per minute, according to a Poisson arrival process. So, for instance, $N(2, 4)$, the number of calls coming in between $t = 2$ and $t = 4$, has Poisson distribution with mean $3 \times (4 - 2) = 6$ and W_3, the waiting time between the second and the third call has Exponential(3) distribution. Compute the following:

a) The probability that no calls arrive between $t = 0$ and $t = 2$.

b) The probability that the first call after $t = 0$ takes more than 2 minutes to arrive.

c) The probability that no calls arrive between $t = 0$ and $t = 2$ and at most four calls arrive between $t = 2$ and $t = 3$.

d) The probability that the fourth call arrives within 30 seconds of the third.

e) The probability that the fifth call takes more than 2 minutes to arrive.

Exercise 4. For any matrix $M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ we define its operator norm by

$$\|M\|_{\mathrm{op}} = \sup_{x\in\mathbb{C}^n, \|x\|_2=1} \|Mx\|_2$$

a) Prove that $\|\cdot\|_{\mathrm{op}} : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to [0,\infty)$ is indeed a norm on the space of $n \times n$ matrices:

i) $\|M\|_{\mathrm{op}} = 0$ if and only if $M = 0$,

ii) $\|\lambda M\|_{\mathrm{op}} = |\lambda|\|M\|_{\mathrm{op}}$ for any $\lambda \in \mathbb{C}$,

iii) $\|M_1 + M_2\|_{\mathrm{op}} \le \|M_1\|_{\mathrm{op}} + \|M_2\|_{\mathrm{op}}$ for any $M_1, M_2 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

b) Prove that $\|M_1M_2\|_{\mathrm{op}} \le \|M_1\|_{\mathrm{op}}\|M_2\|_{\mathrm{op}}$ for any $M_1, M_2 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

c) Prove that if X is an $n \times n$ symmetric matrix with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then $\|X\|_{\mathrm{op}} = \max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\}$.

d) Let $X_n = X_n(\omega)$ be the $n \times n$ GUE ensemble. Describe the distribution of $\|X_n(\omega)\|_{\mathrm{op}}$ as n gets large.

Exercise 5. Let $X \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C})$ be such that $X^T = -X$ (the matrix X is called antisymmetric). The Pfaffian of X is defined by the formula

$$\mathrm{Pf}(X) = \frac{1}{2^n n!} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} X_{\sigma(2i-1),\sigma(2i)}$$

where $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ.

a) Let $J_n = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C})$ (the matrix J_n consists of n 2×2 blocks on the diagonal). Prove that $\mathrm{Pf}(J_n) = 1$.

b) Prove that $\mathrm{Pf}(Y^TXY) = \mathrm{Pf}(X)(\det Y)$ for every $Y \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C})$.

c) Prove that $(\mathrm{Pf}(X))^2 = \det(X)$.