PCMI 2017 - Introduction to Random Matrix Theory
Homework #4 – 06.30.2017

Exercise 1. a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X\}_{n\geq 1}$ and X be random variables on Ω and suppose that $\{X_n\}_{n\geq 1}$ are uniformly bounded (i.e., there exists $M > 0$ such that $|X_n(\omega)| \leq M$ for any $\omega \in \Omega$ and any $n \geq 1$) and X_n converges to X in probability.

(i) Convergence of expectations: $\lim_n \mathbb{E}[X_n] = \mathbb{E}[X]$

(ii) Convergence of variances: $\lim_n \mathrm{Var}[X_n] = \mathrm{Var}[X]$.

(iii) Convergence of the moments: $\lim_n \mathbb{E}[X_n^k] = \mathbb{E}[X^k]$, for any positive integer k. Hint: $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1})$ for any $a, b \in \mathbb{R}$ and any positive integer k.

Exercise 2. Find a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a sequence $\{X_n\}_{n\geq 1}$ of simple random variables on Ω (not uniformly bounded) and a random variable X on Ω such that X_n converges to X in probability, but $\lim_n \mathbb{E}[X_n] \neq \mathbb{E}[X]$ and $\lim_n \mathrm{Var}[X_n] \neq \mathrm{Var}[X]$.

Exercise 3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\{X\}_{n\geq 1}$ and X be random variables on Ω. Let $r \geq 1$. We say that X_n converges to X in r-th mean (or in the L^r-norm) if

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^r] = 0$$

If X_n converges to X in r-th mean (or in the L^r-norm) we write $X_n \xrightarrow{L^r} X$.

When X_n converges the the r-th mean to X for $r = 1$, we say that X_n converges in mean to X.

When X_n converges the the r-th mean to X for $r = 2$, we say that X_n converges in mean square to X.

a) Let $r, s \in \mathbb{R}$ be such that $s > r \geq 1$. Prove that $X_n \xrightarrow{L^s} X$ implies $X_n \xrightarrow{L^r} X$ (convergence in s-the mean implies convergence in r-th mean). Hint: Use Lyapunov's Inequality.

b) Let $r \geq 1$. Prove that $X_n \xrightarrow{L^r} X$ implies $X_n \xrightarrow{P} X$ (convergence in r-the mean implies convergence in probability). Hint: Use Markov's Inequality.

c) Let $r \geq 1$. Prove that convergence in r-th mean does not imply convergence almost everywhere (find a counterexample).

d) Let $r \geq 1$. Prove that convergence almost everywhere does not imply convergence in r-th mean (find a counterexample).

e) Let $r, s \in \mathbb{R}$ be such that $s > r \geq 1$. Prove that convergence in r-th mean does not imply convergence in s-th mean (find a counterexample).

f) Prove that convergence in distribution does not imply convergence in probability (find a counterexample).

Thus, for any real numbers r, s such that $s > r \geq 1$, we have:

$$\begin{array}{ccccc} X_n \xrightarrow{L^s} X & \Rightarrow & X_n \xrightarrow{L^r} X & & \\ & & \Downarrow & & \\ X_n \xrightarrow{\text{a.e.}} X & \Rightarrow & X_n \xrightarrow{P} X & \Rightarrow & X_n \xrightarrow{D} X \end{array}$$

$$\begin{array}{ccc} X_n \xrightarrow{L^r} X & \not\Rightarrow & X_n \xrightarrow{L^s} X \\ X_n \xrightarrow{\text{a.e.}} X & \not\Rightarrow & X_n \xrightarrow{L^r} X \\ X_n \xrightarrow{L^r} X & \not\Rightarrow & X_n \xrightarrow{\text{a.e.}} X \\ X_n \xrightarrow{D} X & \not\Rightarrow & X_n \xrightarrow{P} X \end{array}$$

Exercise 4. The complex normal distribution $N(\mu, \sigma^2)_{\mathbb{C}}$ of mean μ and variance $\sigma^2 > 0$, is given by the density function $f : \mathbb{C} \to \mathbb{R}$ defined by

$$f(z) = \frac{1}{\pi\sigma^2} e^{-\frac{|z-\mu|^2}{\sigma^2}}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{C}$ be a complex random variable with distribution $N(\mu, \sigma^2)_{\mathbb{C}}$. Find the distribution and the density functions of the random variables $\mathrm{Re}X$, $\mathrm{Im}X$, and $|X|$.

Exercise 5. a) Use Mathematica to plot the eigenvalues of matrices sampled from the Bernoulli i.i.d. random matrix ensemble with common coefficient distribution X such that $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = \frac{1}{2}$, for matrices of sizes $n = 10$, $n = 20$, $n = 100$.

b) Use Mathematica to plot the eigenvalues of matrices sampled from the real Gaussian i.i.d. random matrix ensemble with common coefficient distribution $X = N(0, 1)_{\mathbb{R}}$, for matrices of sizes $n = 10$, $n = 20$, $n = 100$.

c) Use Mathematica to plot the eigenvalues of matrices sampled from the complex Gaussian i.i.d. random matrix ensemble with common coefficient distribution $X = N(0, 1)_{\mathbb{C}}$, for matrices of sizes $n = 10$, $n = 20$, $n = 100$.

Exercise 6. a) Use Mathematica to plot the eigenvalues of matrices sampled from the symmetric Bernoulli random matrix ensemble with common coefficient distribution X such that $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = \frac{1}{2}$, for matrices of sizes $n = 10$, $n = 20$, $n = 100$.

b) Use Mathematica to plot the eigenvalues of matrices sampled from the Gaussian Orthogonal Ensemble ($X_{ij} = N(0,1)_{\mathbb{R}}$ for $i < j$ and $X_{ii} = N(0,2)_{\mathbb{R}}$), for matrices of sizes $n = 10$, $n = 20$, $n = 100$.

Exercise 7. Use Mathematica to plot the eigenvalues of matrices sampled from the Gaussian Unitary Ensemble ($X_{ij} = N(0,1)_{\mathbb{C}}$ for $i < j$ and $X_{ii} = N(0,1)_{\mathbb{R}}$), for matrices of sizes $n = 10$, $n = 20$, $n = 100$.