PCMI 2017 - Introduction to Random Matrix Theory
Homework #3 – 06.29.2017

Exercise 1. Show that if U has uniform $[0,1]$ distribution, then the random variable $Y = \tan(\pi U - \frac{\pi}{2})$ has the Cauchy distribution.

Exercise 2. Let X, Y, Z be independent identically distributed random variables with the uniform $[0,1]$ distribution. Compute the probability that the polynomial $P(t) = Xt^2 + Yt + Z$ has two distinct real roots.

Exercise 3. Let $X = X(\omega)$ be a random 2×2 matrix with independent identically distributed coefficients X_{ij}, $1 \leq i, j \leq 2$ and such that each random variable X_{ij} is Bernoulli distributed with values in the set $\{-1, 1\}$ and such that $\mathbb{P}(X_{ij} = -1) = \frac{1}{2}$ and $\mathbb{P}(X_{ij} = 1) = \frac{1}{2}$. Find the distribution functions of $\operatorname{tr}(X)$, $\det(X)$ (the trace and the determinant of the random matrix X). Describe the distribution of the eigenvalues of the matrix X.

Exercise 4. Let X and Y be two independent random variables. Prove that $\phi_{X+Y}(t) = \phi_X(t)\, \phi_Y(t)$.

Exercise 5. Let X be a random variable, $a, b \in \mathbb{R}$ and $Y = aX + b$. Prove that $\phi_Y(t) = e^{itb} \phi_X(at)$.

Exercise 6. If $Y = N(\mu, \sigma^2)$ (the random variable with mean μ and variance σ^2), then $\phi_Y(t) = \mathbb{E}(e^{itY}) = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$. In particular, the characteristic function of the standard normal random variable $N(0,1)$ is $e^{-\frac{t^2}{2}}$.