PCMI 2017 - Introduction to Random Matrix Theory
Handout #11 – 07.13.2017

THE CIRCULAR BETA ENSEMBLE

Definition (The Circular Beta Ensemble) The circular beta ensemble is an ensemble of $n \times n$ random unitary matrices such that if their eigenvalues are $\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\}$ (with $0 \leq \theta_k < 2\pi$ for any k), then the probability density function of the phases θ_k is given by

$$f(\theta_1, \ldots, \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < j \leq n} |e^{i\theta_k} - e^{i\theta_j}|^\beta$$

The CMV Matrix. Given an infinite sequence of coefficients $\alpha_0, \alpha_1, \ldots$ in $\mathbb{D}$ (the open unit disk in $\mathbb{C}$), we define 2×2 matrices

$$\Theta_k = \begin{pmatrix} \overline{\alpha_k} & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$. From these, we form block-diagonal matrices

$$\mathcal{L} = \operatorname{diag}\big(\Theta_0, \Theta_2, \Theta_4, \ldots\big) \quad \text{and} \quad \mathcal{M} = \operatorname{diag}\big(\Theta_{-1}, \Theta_1, \Theta_3, \ldots\big),$$

where $\Theta_{-1} = [1]$, the 1×1 identity matrix.

The matrix $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots) = \mathcal{L}\mathcal{M}$ is unitary. The matrix $\mathcal{C}$ is called the CMV matrix associated to the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots$

The $n \times n$ truncation of $\mathcal{C}$ with $\alpha_{n-1} = e^{i\eta}$ boundary condition is the $n \times n$ upper left corner of $\mathcal{C}$

$$\mathcal{C}^{(n)} = \mathcal{C}(\alpha_0, \ldots, \alpha_{n-2}, e^{i\eta}).$$

Theorem. Given $\beta > 0$, let $\alpha_k \sim \Theta_{\beta(k+1)+1}$ be independent random variables. Suppose that $\alpha_{n-1} = e^{i\eta}$ is uniformly distributed on the unit circle. Then the eigenvalues of $\mathcal{C}^{(n)}$ are distributed according to the circular beta ensemble.

(See Homework #11 for the definition and the properties of the Θ_ν distribution.)