THE CIRCULAR BETA ENSEMBLE

Definition (The Circular Beta Ensemble) The circular beta ensemble is an ensemble of \( n \times n \) random unitary matrices such that if their eigenvalues are \( \{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}\} \) (with \( 0 \leq \theta_k < 2\pi \) for any \( k \)), then the probability density function of the phases \( \theta_k \) is given by

\[
f(\theta_1, \ldots, \theta_n) = \frac{1}{Z_{n,\beta}} \prod_{1 \leq k < j \leq n} |e^{i\theta_k} - e^{i\theta_j}|^\beta
\]

The CMV Matrix. Given an infinite sequence of coefficients \( \alpha_0, \alpha_1, \ldots \) in \( \mathbb{D} \) (the open unit disk in \( \mathbb{C} \)), we define \( 2 \times 2 \) matrices

\[
\Theta_k = \begin{pmatrix} \overline{\alpha_k} & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix}
\]

where \( \rho_k = \sqrt{1 - |\alpha_k|^2} \). From these, we form block-diagonal matrices

\[
\mathcal{L} = \text{diag}(\Theta_0, \Theta_2, \Theta_4, \ldots) \quad \text{and} \quad \mathcal{M} = \text{diag}(\Theta_{-1}, \Theta_1, \Theta_3, \ldots),
\]

where \( \Theta_{-1} = [1] \), the \( 1 \times 1 \) identity matrix.

The matrix \( \mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots) = \mathcal{L}\mathcal{M} \) is unitary. The matrix \( \mathcal{C} \) is called the CMV matrix associated to the Verblunsky coefficients \( \alpha_0, \alpha_1, \ldots \).

The \( n \times n \) truncation of \( \mathcal{C} \) with \( \alpha_{n-1} = e^{i\eta} \) boundary condition is the \( n \times n \) upper left corner of \( \mathcal{C} \)

\[
\mathcal{C}^{(n)} = \mathcal{C}(\alpha_0, \ldots, \alpha_{n-2}, e^{i\eta}).
\]

Theorem. Given \( \beta > 0 \), let \( \alpha_k \sim \Theta_{\beta(k+1)+1} \) be independent random variables. Suppose that \( \alpha_{n-1} = e^{i\eta} \) is uniformly distributed on the unit circle. Then the eigenvalues of \( \mathcal{C}^{(n)} \) are distributed according to the circular beta ensemble.

(See Homework #11 for the definition and the properties of the \( \Theta_\nu \) distribution.)