PCMI 2017 - Introduction to Random Matrix Theory
Handout #8 – 07.07.2017

THE TRACY-WIDOM DISTRIBUTION

Recall (from Homework #5) the Airy function $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ defined by:

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$$

Let $u(x)$ be the solution of the Painlevé II equation Painlevé equation $u''(x) = 2(u(x))^3 + xu(x)$ with $u(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$.

We say that a random variable X has Tracy-Widom distribution if its distribution function is

$$F(t) = e^{-\frac{1}{2}\int_t^\infty (x-t)u^2(x)\,dx}$$

We define new distribution functions:

$$F_2(t) = F(t)^2$$
$$F_1(t) = e^{-\frac{1}{2}\int_t^\infty u(x)\,dx}(F_2(t))^{1/2}$$

Theorem (Tracy-Widom). Let X_n be an $n \times n$ GUE matrix and let $\lambda_{\max}(n)$ be its largest eigenvalue. Then

$$n^{1/6}(\lambda_{\max}(n) - 2\sqrt{n}) \xrightarrow{D} F_2$$

A similar result holds for GOE (replace F_2 with F_1).