Spectral Analysis of New Classes of Deterministic and Random Orthogonal Polynomials on the Unit Circle

by

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Abstract

We present the general theory for orthogonal polynomials on the unit circle (OPUC), which is the study of non-trivial probability measures on the unit circle and the related mathematical objects. We review some classic examples of OPUC and characterize new classes of orthogonal polynomials on the unit circle. We then prove properties relating random Bernoulli Verblunsky coefficients, the zeros of the orthogonal polynomials, and moments of the measure.

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All errors are my own. I can be reached at ay3@williams.edu.

Contents

1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in $\mathbb{C}$ and let μ be a probability measure on the unit circle $\partial\mathbb{D} = \{z : |z| = 1\}$. A point $z \in \partial\mathbb{D}$ is parametrized by $z = e^{i\theta}$. We call the measure μ *trivial* if it is supported on a finite subset of $\partial\mathbb{D}$ and *non-trivial* otherwise. In this thesis we focus on non-trivial probability measures on the unit circle. The functions $\{1, z, z^2, \ldots\}$ are linearly independent in the Hilbert space $L^2(\partial\mathbb{D}, d\mu)$ because μ is non-trivial, implying that no nonzero polynomial vanishes a.e. with respect to μ. Therefore, we can apply Gram-Schmidt to $\{1, z, z^2, \ldots\}$ to construct $\{\Phi_n(z; d\mu)\}$, the monic orthogonal polynomials on the unit circle. The Φ_n's obey:

$$\Phi_n(z) = z^n + \text{lower order terms} \tag{1.1}$$

$$\langle z^n, \Phi_n(z)\rangle = \int e^{-ik\theta}\Phi_n(e^{i\theta})d\mu(\theta) = 0 \qquad k = 0, 1, 2, \ldots n-1, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product defined by $\langle f, g\rangle = \int_0^{2\pi} \overline{f(e^{i\theta})}g(e^{i\theta})d\mu(\theta)$. The monic polynomials are orthogonal; that is, $\langle \Phi_n, \Phi_m\rangle = 0$ if $n \neq m$. We can also define $\{\varphi_n\}$ the set of orthonormal polynomials by

$$\varphi_n = \frac{\Phi_n}{||\Phi_n||} \tag{1.3}$$

where $||\cdot||$ is the L^2 norm. The monic polynomials obey the following recursion relation:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \tag{1.4}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z), \tag{1.5}$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(\frac{1}{\bar{z}})}$ is called the *reverse polynomial of* Φ_n and α_n is called the Verblunsky coefficient. Given any non-trivial probability measure μ on the unit circle, there exists a bijection between μ, the orthogonal polynomials, the list of Verblunsky coefficients $\{\alpha_i\}_{i=0}^{\infty}$, and other related mathematical objects.

Orthogonal polynomials on the real line (OPRL) have been studied extensively by analysts because they play a central role in the study of inverse spectral problems and the moment problem. Similarly, orthogonal polynomials on the unit circle (OPUC) have important applications to fields such as linear prediction and filtering theory, and OPUC serves as an analogy to the spectral theory of OPRL (e.g. the Discrete Schrödinger Operator). Moreover, OPUC itself is an intrinsically fascinating mathematical theory to study. As Simon note in his exposition (Simon 2005*a*), "Significant missing materials (in the study of OPUC) involve some explicit examples". Therefore, we contribute to the literature by studying and characterizing new classes of orthogonal polynomials on the unit circle. Furthermore, we prove new results on discrete random Verblunsky coefficients.

This thesis is organized as follows. Section 2 provides the general theory for analyzing measures on the unit circle. Most importantly, we show how measures can be decomposed generally and how a measure on the unit circle can be recovered from analytic functions within the open unit disk. Section 3 presents the formal theory of OPUC. Section 4 contains explicit examples for OPUC. Section 4.1 has four examples from Simon's book (Simon 2005*b*) and Section 4.2 is where we present the new examples we study. Finally, Section 5 contains theorems relating random Bernoulli Verblunsky coefficients and the properties of the zeros of the (random) orthogonal polynomials.

2 General Analysis of Measures

Central to orthogonal polynomials on the unit circle is the study of non-trivial probability measures on the unit circle. That is, positive measures μ on $\partial\mathbb{D}$ with

$$\mu(\partial\mathbb{D}) = 1.$$

This section reviews the theoretical background for decomposing general measures and presents the tools for analyzing positive measures on the unit circle. The reader is directed to Rudin (1987) and Folland (2005) for more detailed expositions.

2.1 Decomposition of Measures

A *pure point* of a measure $d\mu$ on the unit circle is a point $z_0 \in \partial\mathbb{D}$ with $\mu(z_0) > 0$. The restriction of μ to the set of all pure points, P, is called the *pure point* part of $d\mu$. The Lebesgue-Radon-Nikodym decomposition theorem allows $d\mu$ to be decomposed into

$$d\mu \quad = \quad w(\theta)\frac{d\theta}{2\pi} + d\mu_s,$$

where $w(\theta)\frac{d\theta}{2\pi}$ is called the *absolutely continous* part (with respect to Lebesgue) of $d\mu$ and $w \in L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$. The measure $d\mu_s$ is mutually singular with respect to $\frac{d\theta}{2\pi}$. The restriction of the singular part to the unit circle minus the pure points, $\mu_s \restriction (\partial\mathbb{D}\backslash P)$, is the *singular continuous part* of $d\mu$. The *support* of μ is

$$\text{supp}(d\mu) = \overline{\{z|z \in \partial\mathbb{D}|\mu(z) > 0\}}.$$

Next we review the Lebesgue-Radon-Nikodym decomposition theorem. First, we need a few definitions.

Definition 2.1. (Signed Measure) Let (X, Σ) be a measurable space. A function $\nu : \Sigma \to [-\infty, \infty]$ is called a **signed measure** if:

- $\nu(\emptyset) = 0$;
- ν assumes at most one of the values $\pm\infty$;
- if $\{E_j\}$ is a sequence of disjoint sets in Σ, then $\nu(\cup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$, where the latter sum converges absolutely if $\nu(\cup_1^\infty E_j)$ is finite.

Definition 2.2. (Mutually Singular) Let μ and ν be two signed measures. We say that μ and ν are **mutually singular** (denoted $\mu \perp \nu$) if there exist $E, F \in \Sigma$ such that $E \cap F = \emptyset$, $E \cup F = X$, E is null for μ, and F is null for μ.

Definition 2.3. (Absolutely Continuous) Let ν be a signed measure and μ be a positive measure on (X, Σ). We say that ν is **absolutely continous** with respect to μ (denoted as $\nu \ll \mu$) if for every $E \in \Sigma$ for which $\mu(E) = 0$, we have that $\nu(E) = 0$. It can be shown that this definition is equivalent to: for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

The next two lemmas are needed in order to prove the Lebesgue-Radon-Nikodym theorem. The reader is directed to Folland (2005) for the proof.

Lemma 2.4. *If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$. That is, $\int_E f d\mu \ll \mu$.*

From here on, to express the relationship $\nu(E) = \int_E f d\mu$ for any $E \in \Sigma$, we use the notation $d\nu = f d\mu$.

Lemma 2.5. *Suppose that ν and μ are finite measures on (X, Σ). Either $\nu \perp \mu$, or there exist $\epsilon > 0$ and $E \in \Sigma$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on E.*

Theorem 2.6. *(Lebesgue-Radon-Nikodym)*

Let ν be a σ-finite signed measure and μ a σ-finite positive measure on (X, Σ). There exist unique σ-finite signed measures λ, ρ on (X, Σ) such that

$$\lambda \perp \mu, \qquad \rho \ll \mu, \qquad \text{and} \qquad \nu = \lambda + \rho;$$

λ is the mutually singular part of ν with respect to μ and ρ is the absolutely continuous part. Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$, and any two such functions are equal μ-a.e.

Proof. Case I: Suppose that ν and μ are finite, positive measures. First, we construct the set

$$\mathcal{F} = \{f : X \to [0, \infty] : \int_E f d\mu \leq \nu(E) \quad \text{for all} \quad E \in \Sigma\}.$$

This means that $\mathcal{F}$ contains all the positive functions whose μ integral over any subset E of Σ is bounded above by $\nu(E)$. Note that $\mathcal{F}$ is non-empty because the zero function is in $\mathcal{F}$. Also, observe that $\mathcal{F}$ is closed under taking the maximum of two functions. To see this, if $f, g \in \mathcal{F}$ and $h = \max(f, g)$, let $A = \{x : f(x) > g(x)\}$. Then, for all $E \in \Sigma$,

$$\int_E h d\mu = \int_{E \cap A} f d\mu + \int_{E \setminus A} g d\mu \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E).$$

Let $a = \sup\{\int f d\mu : f \in \mathcal{F}\}$. Since $a \leq \nu(X) < \infty$ by our construction of $\mathcal{F}$ and our assumption that ν is a finite measure, we may choose a sequence

$\{f_n\} \subset \mathcal{F}$ such that $\int f_n d\mu \to a$. Let $g_n = \max(f_1, f_2, ..., f_n)$ and $f = \sup_n f_n$. We know that $g_n \in \mathcal{F}$ (closure under taking maximum), g_n increases pointwise to f (fixing x, they are the same), and $\int g_n d\mu \geq \int f_n d\mu$ (g_n is the maximum). By the definition of a and that $\int f_n d\mu \to a$, we know that $\lim \int g_n d\mu = a$. Since $\{g_n\}$ is a set of increasing functions, by the Monotone Convergence Theorem, we have that $\lim \int g_n d\mu = \int \lim g_n d\mu = \int f d\mu = a$ and $f \in \mathcal{F}$.

We have finished the construction of f such that $d\rho = f d\mu$ and $\rho \ll \mu$. Now we claim that $d\lambda = d\nu - f d\mu$ is mutually singular with respect to μ. First, observe that $d\lambda$ is positive because $f \in \mathcal{F}$. Suppose for contradiction that λ is not mutually singular with μ. By Lemma 2.5, there exist $E \in \Sigma$ and $\epsilon > 0$ such that $\mu(E) > 0$ and $\lambda \geq \epsilon\mu$ on E. But then

$$\epsilon \chi_E d\mu \leq d\lambda = d\nu - f d\mu$$

$$(f + \epsilon\chi_E) d\mu \leq d\nu$$

For our fixed $\epsilon > 0$ and $E \in \Sigma$, we have that

$$\int_E (f + \epsilon\chi_E) d\mu \leq \nu(E).$$

Observe that the inequality above is true for integrating over any $A \in \Sigma$, as

$$\begin{aligned}
\int_A (f + \epsilon\chi_E) d\mu &= \int_{A\cap E} (f + \epsilon\chi_E) d\mu + \int_{A\backslash E} (f + \epsilon\chi_E) d\mu \\
&= \int_{A\cap E} (f + \epsilon\chi_E) d\mu + \int_{A\backslash E} f d\mu \\
&\leq \nu(A \cap E) + \nu(A\backslash E) = \nu(A),
\end{aligned}$$

where in the second line, since $\lambda \leq \epsilon\mu$ everywhere on E, it must be also true on $(A \cap E) \subset E$. Since the above is true for any $A \in \Sigma$, by definition, we have that $(f + \epsilon\chi_E) \in \mathcal{F}$. It follows that

$$\int (f + \epsilon\chi_E) d\mu = a + \epsilon\mu(E) > a$$

contradicting the definition of a.

Hence, we have shown the existence of λ, f, and $d\rho = f d\mu$. To show uniqueness, suppose that there also exists $d\nu = d\lambda' + f' d\mu$. It follows that $d\lambda - d\lambda' = (f' - f)d\mu$. From Lemma 2.5 we have that $\lambda - \lambda' \perp \mu$ and $(f' - f)d\mu \ll d\mu$. This can only be true if $d\lambda - d\lambda = (f' - f)d\mu = 0$. Hence, $\lambda = \lambda'$, $f = f'$ and we are done with uniqueness. This concludes the case when μ and ν are finite, positive measures.

Case II: Suppose that μ and ν are instead σ-finite, positive measures. Then X can be broken up into a countable disjoint union of μ-finite sets and also a countable disjoint union of ν-finite sets. Next, we may take the intersection of all these sets to obtain $X = \cup_{j=1}^{\infty} A_j$, where $\{A_j\}$ is a disjoint sequence whose elements are both ν-finite and μ-finite. For any $E \in \Sigma$, we define $\mu_j(E) = \mu(E \cap A_j)$ and $\nu_j(E) = \nu(E \cap A_j)$. From Case I, for every j, we may construct $d\nu_j = d\lambda_j + f_j d\mu_j$ with the desired properties. Since $\mu_j(A_j^C) = \nu_j(A_j^C) = 0$, it follows that we may assume $f_j = 0$ on A_j^C. Let $\lambda = \sum \lambda_j$ and $f = \sum f_j$. It follows from Lemma 2.5 that $d\nu = d\lambda + f d\mu$, $\lambda \perp \mu$, and $d\lambda$ and $f d\mu$ are σ-finite. Uniqueness is the same proof as before.

Generally, if ν is a signed measure, we may apply the preceding argument to the positive and negative parts separately and combine them in the end. □

2.2 Poisson Representation

This section shows the general theory for how a measure μ on $\partial\mathbb{D}$ can be recovered from an analytic function on $\mathbb{D}$ through the Poisson Representation and non-tangential limit.

Definition 2.7. (Poisson Kernel) The Poisson Kernel is the function

$$P_r(t) \;=\; \sum_{-\infty}^{\infty} r^{|n|} e^{int} \qquad (0 \leq r < 1). \tag{2.1}$$

Let $z = re^{i\theta}$. It is a straight-forward calculation to show that

$$P_r(\theta - t) = \mathrm{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] = \frac{1 - r^2}{r^2 - 2r\cos(\theta - t) + 1}.$$

It then follows from (2.11) that for $(0 \le r < 1)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt = 1.$$

Definition 2.8. (Poisson Representation) Let $f \in L^1(\partial\mathbb{D})$. The function $F(re^{i\theta})$ is defined on $\mathbb{D}$ by

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)f(t)dt; \tag{2.2}$$

F defined on $\mathbb{D}$ is called the *Poisson Representation of f*, which we write as $F = P[f]$.

When $P_r(\theta - t)$ is viewed as a function of $z = re^{i\theta}$ and e^{it}, it can be shown that

$$P_r(\theta - t) = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

With this parametrization, consider the Poisson representation of a measure μ on $\partial\mathbb{D}$:

$$P[d\mu] = \int_{\partial\mathbb{D}} P(z, e^{it})d\mu = \int \frac{1 - |z|^2}{|e^{it} - z|^2} d\mu(e^{it}).$$

The following theorem shows that if f is a continuous function defined on $\partial\mathbb{D}$, then its Poisson representation behaves nicely near the boundary of $\mathbb{D}$. The reader is directed to Rudin (1987) for the proofs in this section.

Theorem 2.9. *Let $f \in C(\partial\mathbb{D})$ and let Hf be defined on $\mathbb{D} \cup \partial\mathbb{D}$ by:*

$$Hf(re^{i\theta}) = \begin{cases} f(re^{i\theta}) & \text{if } r = 1 \\ P[f] & \text{if } 0 \le r < 1. \end{cases}$$

It follows that $Hf \in C(\partial\mathbb{D})$.

The goal is to recover the value of $f \in L^1(\partial\mathbb{D})$ from its Poisson representation $F = P[f]$. We cannot approach the boundary $\partial\mathbb{D}$ tangentially because the Poisson integral is undefined there. Instead, we approach the boundary within the open unit disk $\mathbb{D}$, which motivates the definition of a non-tangential limit.

For $0 < \alpha < 1$, define Ω_α to be the union of the disk centered at 0 with radius α and all the line segments from $z = 1$ to the points within that disk. Pictorially, Ω_α is a cone with vertex at 1. Near $z = 1$, Ω_α has an angle of opening 2θ, where $\alpha = \sin(\theta)$. Because approaching the vertex 1 within Ω_α cannot be tangent to $\partial\mathbb{D}$, we call Ω_α a non-tangential approach region, with vertex 1. We denote the approach region of another point $e^{i\theta}$ on $\partial\mathbb{D}$ by $e^{i\theta}\Omega_\alpha$.

Definition 2.10. (Non-tangential limits) A function F, defined in $\mathbb{D}$, has a non-tangential limit L at $e^{i\theta} \in \partial\mathbb{D}$ if for all $\alpha < 1$,

$$\lim_{j\to\infty} F(z_j) = L$$

for every sequence $\{z_j\}$ that converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$.

Lemma 2.11. *If μ is a positive Borel measure on $\partial\mathbb{D}$ and $(D\mu)(e^{i\theta_0}) = 0$ at θ_0, then its Poisson integral $P[d\mu]$ has non-tangential limit 0 at $e^{i\theta_0}$. In other words, if the Radon-Nikodym derivative at a point on the unit circle is zero, then the non-tangential limit of the Poisson Kernel (from $d\mu$) at that point is zero.*

Definition 2.12. (Lebesgue Point) The point $e^{i\theta_0}$ is a *Lebesgue point* for the function $f : \partial\mathbb{D} \to \mathbb{C}$ if

$$\lim_{r\to 0} \frac{\int_{I_r} |f(e^{i\theta}) - f(e^{i\theta_0})| d\theta}{r} = 0,$$

where I_r is the arc of length r centered at $e^{i\theta}$.

Lemma 2.13. *If $f \in L^1(\partial\mathbb{D})$, then $P[f]$ has non-tangential limit $f(e^{i\theta})$ at every Lebesgue point $e^{i\theta}$ of f.*

The following theorem allows to recover μ from analytic functions on $\mathbb{D}$ through the Poisson Representation and non-tangential limits.

Theorem 2.14. *Let $d\mu = f dt + d\mu_s$ be the Lebesgue-Radon-Nikodym decomposition of the measure μ on $\partial\mathbb{D}$, where $f \in L^1(\partial\mathbb{D})$ and μ_s is the mutually* *singular part. Then the Poisson Integral $P[d\mu]$ has non-tangential limit $f(e^{i\theta})$ at almost all points of $\partial\mathbb{D}$.*

Proof. Apply Lemmas 2.11 and 2.13. □

2.3 Carathédory and Schur Functions

The *Carathédory function* F of μ on $\partial\mathbb{D}$ is defined on the open unit disk $\mathbb{D}$ by

$$F(z) \;=\; \int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta); \tag{2.3}$$

F is an analytic function on $\mathbb{D}$ and it satisfies

$$F(0) \;=\; 1 \tag{2.4}$$

$$\mathrm{Re} F(z) \;>\; 0. \tag{2.5}$$

The *Schur function* f is then defined through the bijective map

$$F(z) \;=\; \frac{1+zf(z)}{1-zf(z)} \tag{2.6}$$

$$f(z) \;=\; \frac{1}{z}\frac{F(z)-1}{F(z)+1}; \tag{2.7}$$

f is an analytic function from $\mathbb{D}$ to $\overline{\mathbb{D}}$, so it satisfies

$$\sup_{z\in\mathbb{D}} |f(z)| \leq 1.$$

Since $|f| \leq 1$, we have the following universal bound on Carathédory functions for every point $z \in \mathbb{D}$:

$$\frac{1-|z|}{1+|z|} \leq |F(z)| \leq \frac{1+|z|}{1-|z|}.$$

We show here that (2.3) sets up a bijection correspondence between probability measures μ on $\partial\mathbb{D}$ and analytic functions satisfying (2.4) and (2.5). Let

$\{c_n\}_{n=-\infty}^{\infty}$ be the moments of μ as defined by

$$c_n = \int e^{-in\theta} d\mu(\theta). \tag{2.8}$$

Let F be a Carathédory function defined from its Taylor series coefficients by

$$F(z) \;=\; 1 + 2\sum_{n=1}^{\infty} c_n z^n.$$

Define the measure $d\mu_r(\theta)$ as

$$d\mu_r(\theta) = \mathrm{Re}F(re^{i\theta})\frac{d\theta}{2\pi}.$$

For $n = 0, \pm 1, \pm 2, \ldots$, the moments of $d\mu_r(\theta)$ are

$$\int e^{-in\theta} d\mu_r(\theta) = \int e^{-in\theta}\left(1 + \sum_{j=1} c_n r^j e^{ij\theta} + \overline{c}_n r^j e^{-ij\theta}\right)\frac{d\theta}{2\pi} = c_n r^{|n|}, \tag{2.9}$$

as the $\int e^{in\theta}\frac{d\theta}{2\pi} = 0$ for $|n| \geq 1$. Equation (2.9) essentially computes the integral for any Laurent polynomial $h(z)$ (a linear combination of positive and negative integer powers of z). Hence, taking $r \uparrow 1$, $\int h(\theta) d\mu_r(\theta)$ has a limit. Since the Laurent polynomials are dense in $C(\partial\mathbb{D})$ and the integral can be bounded above by $||h||_\infty$, the $d\mu_r(\theta)$'s converge weakly to the measure $d\mu$ whose moments are

$$e^{-in\theta} d\mu(\theta) = c_n.$$

Furthermore, the Poisson representation for $F(rz)$, $0 \leq r < 1$ is

$$F(rz) \;=\; \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_r(\theta).$$

Because $\frac{e^{i\theta}+z}{e^{i\theta}-z}$ is continuous, the weak convergence implies

$$F(z) \;=\; \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu \tag{2.10}$$

for a unique probability measure μ on $\partial\mathbb{D}$. This proves the bijection between F and $d\mu$, which can be summarized by the following lemma.

Lemma 2.15. *We have* $\frac{1}{2\pi}ReF(re^{i\theta})d\theta$ *converges weakly to* $d\mu$.

Finally, the following theorem shows how the a.c. part of the measure μ can be recovered from its associated Carathédory function.

Theorem 2.16. *Define* $F(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta})$*, which exists for Lebesgue a.e.* θ*. Let the Lebesgue-Radon-Nikodym decomposition of* $d\mu$ *be* $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$*. Then*

$$w(\theta) \quad = \quad ReF(e^{i\theta}).$$

Proof. From the previous lemma, we know that $d\mu_r(\theta) = \mathrm{Re}F(re^{i\theta})\frac{d\theta}{2\pi}$ converges weakly to $d\mu$ as $r \uparrow 1$. Taking the non-tangential limit as $r \uparrow 1$, this theorem follows from Theorem 2.14. □

Combined with (2.6) and (2.7), we have

$$\begin{aligned} \mathrm{Re}F(z) &= \frac{1-|zf|^2}{|1-zf(z)|^2} \\ d\mu_{a.c}(\theta) &= \frac{1-|f(e^{i\theta})|^2}{|1-e^{i\theta}f(e^{i\theta})|^2}\frac{d\theta}{2\pi}. \end{aligned}$$

This shows that the a.c. part of the measure is supposed on $\{\theta||f(e^{i\theta})| < 1\}$. Furthermore, Geronimus showed that Schur functions are also defined by their *Schur parameters*, which is a list of coefficients $\{\gamma_n\}_{n=0}^{\infty}$. The Schur parameters obey $|\gamma_i| < 1$ if μ is non-trivial. The Schur function f can be written as a continued fraction expansion in terms of its Schur parameters by

$$f \quad = \quad \gamma_0 + \frac{1-|\gamma_0|^2}{\overline{\gamma_0} + \frac{1}{z\gamma_1 + \frac{z(1-|\gamma_1|^2)}{z\gamma_2+\ldots}}}. \tag{2.11}$$

Due to Geronimus' Theorem (see Simon (2005b) for the proof), we in fact have that $\{\gamma_n\}_{n=0}^{\infty} = \{\alpha_n\}_{n=0}^{\infty}$. Namely, the Schur parameters are equal to the Verblunsky coefficients for OPUC.

2.4 Szegö Function

The *Szegö condition* holds for $w(\theta)$ when

$$\int_0^{2\pi} \log(w(\theta)) \frac{d\theta}{2\pi} > -\infty. \tag{2.12}$$

Definition 2.17. (Szegö function) Suppose that the Szegö condition holds for w. Then the Szegö function, $D(z)$ is defined by

$$D(z) \quad = \quad \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(w(\theta))d\theta\right).$$

Furthermore, if the Szegö condition holds, then

(i) $\lim_{n\to\infty}\varphi_n^*(z) = D(z)^{-1}$, and

(ii) for Lebesgue a.e. $\theta \in [0, 2\pi)$, $\lim_{r\uparrow 1} D(re^{i\theta}) = D(e^{i\theta})$ exists and

$$|D(e^{i\theta})|^2 = w(\theta).$$

3 Verblunsky Coefficients and the Szegö Recurrence

This section presents the formal theory of orthogonal polynomials on the unit circle (OPUC).

3.1 Orthogonal Polynomials and Reverse Polynomials

Let μ be a non-trivial probability measure on $\partial\mathbb{D} = \{z : |z| = 1\}$, the unit circle. That is, $\mu(\partial\mathbb{D}) = 1$ and μ is positive. Let $\Phi_n(z; d\mu)$ be the monic orthogonal polynomials (monic OP's) constructed from Gram-Schmidt in $L^2(\partial\mathbb{D}, d\mu)$ and $\varphi_n(z; d\mu)$ the orthonormal polynomials, where $\varphi_n = \frac{\Phi_n}{||\Phi_n||}$. Define

$$\kappa_n = ||\Phi_n||^{-1}.$$

It follows that $\varphi_n = \kappa_n z^n +$ lower order terms. Given any monic polynomial P of degree n, $(\Phi_n - P) \perp \Phi_n$ by the Gram-Schmidt construction. Therefore,

$$\begin{aligned} \langle \Phi_n, \Phi_n - P \rangle &= 0 \\ \langle \Phi_n, P \rangle &= \langle \Phi_n, \Phi_n \rangle = ||\Phi_n||^2. \end{aligned}$$

Applying the relation above to $P = z^n$ and $P = z\Phi_{n-1}$, we have

$$||\Phi_n||^2 = \langle z^n, \Phi_n \rangle = \langle z\Phi_{n-1}, \Phi_n \rangle = \kappa_n^{-2}.$$

Furthermore, $z^{-1} = \overline{z}$ for $z \in \partial\mathbb{D}$, so

$$\begin{aligned} ||z\Phi_n|| &= ||\Phi_n|| \\ \langle zf, g \rangle &= \langle f, z^{-1}g \rangle, \end{aligned}$$

where the second equality follows from the definition of an inner product in L^2. (For more on the functional analysis background, the reader is directed to Teschl (2017).) On $L^2(\partial\mathbb{D}, \partial\mu)$, define the reverse mapping $R_n : L^2 \to L^2$ by

$$(R_n f)(z) = z^n \overline{f(z)}.$$

To see that R_n is an anti-unitary map, observe

$$\begin{aligned}\langle R_n f, R_n g\rangle &= \langle z^n\overline{f}, z^n\overline{g}\rangle \\ &= \int_{\partial\mathbb{D}} \overline{z^n\overline{f}} z^n\overline{g} d\mu \\ &= \int_{\partial\mathbb{D}} f\overline{g} d\mu \\ &= \langle g, f\rangle \\ &= \overline{\langle f, g\rangle}.\end{aligned}$$

Since R_n is anti-unitary, it follows that it is a bijection. In fact, because $\overline{z^n} = z^{-n}$ for $z \in \partial\mathbb{D}$, $R_n^2 = I$ (i.e. $R_n^{-1} = R_n$). Also, note that R_n is an anti-linear map (i.e., $R_n(\alpha f) = \overline{\alpha} R_n(f)$). To see this,

$$\begin{aligned}R_n(\alpha f) &= z^n\overline{\alpha f} \\ &= \overline{\alpha}(z^n\overline{f}) \\ &= \overline{\alpha} R_n(f).\end{aligned}$$

When R_n is applied to a degree n polynomial, the coefficients are reversed with complex conjugation. To see this, let $P(z) = \sum_{j=0}^n c_j z^j$ ($c \in \mathbb{C}$) be a degree n polynomial on the unit circle ($z \in \partial\mathbb{D}$). Then

$$\begin{aligned}R_n(P(z)) &= R_n(\sum_{j=0}^n c_i z^j) \\ &= \sum_j \overline{c_j} R_n(z^j) \\ &= \sum_j \overline{c_j} z^{n-j} \\ &= \sum_{k=0}^n \overline{c_{n-k}} z^j.\end{aligned}$$

This property of the reverse map R_n motivates the following definition.

Definition 3.1. (Reverse Polynomial). Let Q_n be a n-degree polynomial.

Define its **reverse polynomial** Q_n^* by

$$Q_n^*(z) \;=\; z^n\overline{Q_n\left(1/\overline{z}\right)}.$$

Since $\frac{1}{\overline{z}} = z$ for $z \in \partial\mathbb{D}$, the definition is indeed consistent with R_n. That is, $R_n(Q_n) = Q_n^*$.

Theorem 3.2. *Up to a multiplicative constant, $\Phi_n^*(z)$ is the unique polynomial of degree (at most) n which is orthogonal to $z, z^2, \ldots, z^n$ (i.e., if $\langle P, z^j\rangle = 0$ for $j = 1, 2, \ldots, n$ and deg(P) $\leq n$, then P is a multiple of Φ_n^*). Moreover,*

$$\Phi^*(0) = 1 \tag{3.1}$$

$$||\Phi_n||^2 = ||\Phi_n^*||^2 = \int \Phi_n^*(e^{i\theta})d\mu(e^{i\theta}). \tag{3.2}$$

Proof. That Φ_n^* is orthogonal to $\{z^j\}_{j=1}^{j=n}$ follows from the fact that Φ_n is the orthogonal to $\{z^k\}_{k=0}^{k=n-1}$. For $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\langle \Phi_n^*(z), z^j\rangle &= \langle R_n\Phi_n, z^j\rangle \\
&= \langle R_n\Phi_n, R_n(R_n z^j)\rangle \\
&= \langle R_n z^j, \Phi_n\rangle \qquad \text{(anti-unitarity)} \\
&= \langle z^n\overline{z^j}, \Phi_n\rangle \\
&= \langle z^{n-j}, \Phi_n\rangle \\
&= \langle z^k, \Phi_n\rangle \qquad k = 0, 1, \ldots, n-1 \\
&= 0 \qquad \text{(property of } \Phi_n\text{)}.
\end{aligned}$$

To prove (3.1) observe that

$$\begin{aligned}
\Phi_n(z) &= z^n + \text{lower order} \\
\Phi_n^*(z) &= R_n\Phi_n(z) \\
&= z^n\overline{z^n + \text{lower order}} \\
&= 1 + \text{higher order (up to } n\text{)}.
\end{aligned}$$

Next, uniqueness of Φ_n^* follows from a simple dimensionality argument. We see that Φ_n^* is non-zero because $\Phi_n^*(0) = 1$. Since in span$(1, z, z^2, \ldots, z^n)$, the orthogonal complement of span$(z, z^2, \ldots, z^n)$ has dimension one, any polynomial of degree n must be a multiple of Φ_n^*. To see the first equality in (3.2),

$$\begin{aligned} ||\Phi_n||^2 &= \langle \Phi_n, \Phi_n \rangle \\ &= \langle R_n\Phi_n, R_n\Phi_n \rangle \\ &= \langle \Phi_n^*, \Phi_n^* \rangle \\ &= ||\Phi_n^*||^2. \end{aligned}$$

To see the second equality in (3.2),

$$\begin{aligned} ||\Phi_n||^2 &= \langle \Phi_n, \Phi_n \rangle \\ &= \langle \Phi_n, z^n \rangle \\ &= \langle R_n z^n, R_n\Phi_n \rangle \\ &= \langle 1, \Phi_n^* \rangle \\ &= \int \Phi_n^* d\mu. \end{aligned}$$

□

3.2 Verblunsky Coefficients

Theorems 3.3 and 3.4 are key starting points to our study of OPUC. As we will see, they set up a bijection between a sequence of complex numbers in the open unit disk $\mathbb{D}$, orthogonal polynomials, and positive Borel measures on the unit circle $\partial\mathbb{D}$.

Theorem 3.3. *(Szegö Recurrence). For any non-trivial probability measure μ on $\partial\mathbb{D}$, we can define $\{\alpha_n\}_{n=0}^{\infty}$ of numbers in the open unit disk $\mathbb{D}$ such that*

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \tag{3.3}$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z), \tag{3.4}$$

which is called the Szegö Recurrence. The $\{\alpha_n\}_{n=0}^{\infty}$ are called the Verblunsky *coefficients. Moreover,*

$$||\Phi_{n+1}||^2 = (1-|\alpha_n|^2)||\Phi_n||^2 \tag{3.5}$$
$$= \prod_{j=0}^{n}(1-|\alpha_j|^2). \tag{3.6}$$

Proof. To show (3.3), it is enough to show the existence and uniqueness of $-\overline{\alpha_n}$. For $k = 1, 2, \ldots, n$, we have

$$\begin{aligned}\langle z\Phi_n, z^k\rangle &= \langle \Phi_n, \overline{z}z^k\rangle \\ &= \langle \Phi_n, z^{k-1}\rangle \\ &= 0.\end{aligned}$$

This shows that $z\Phi_n$ is orthogonal to $z, z^2, \ldots, z^n$. Since $\Phi_{n+1}(z)$ is also orthogonal to $z, z^2, \ldots, z^n$, their difference must also be orthogonal to $z, z^2, \ldots, z^n$. Because Φ_{n+1} and Φ_n are monic polynomials, it follows that $\Phi_{n+1} - z\Phi_n$ is of degree n, so by Theorem 3.2, we have the existence and uniqueness of $-\overline{\alpha_n}$. To obtain (3.4), apply R_{n+1} to (3.3),

$$\begin{aligned}R_{n+1}\Phi_{n+1} &= R_{n+1}(z\Phi_n) - R_{n+1}(\overline{\alpha_n}\Phi_n^*) \\ \Phi_{n+1}^* &= z^{n+1}\overline{z\Phi_n} - z^{n+1}\overline{\overline{\alpha_n}\Phi_n^*} \\ &= z^n\overline{\Phi_n} - \alpha_n z(z^n\overline{\Phi_n^*}) \\ &= \Phi_n^* - \alpha_n z\Phi_n.\end{aligned}$$

To obtain (3.5), rearrange (3.3) to $z\Phi_n = \Phi_{n+1} + \overline{\alpha_n}\Phi_n^*$ and apply the inner product to both sides. On the right hand side, we have

$$\begin{aligned}&\langle \Phi_{n+1} + \overline{\alpha_n}\Phi_n^*, \Phi_{n+1} + \overline{\alpha_n}\Phi_n^*\rangle \\ =\ &\langle \Phi_{n+1}, \Phi_{n+1}\rangle + \langle \Phi_{n+1}, \overline{\alpha_n}\Phi_n^*\rangle + \langle \overline{\alpha_n}\Phi_n^*, \Phi_{n+1}\rangle + \langle \overline{\alpha_n}\Phi_n^*, \overline{\alpha_n}\Phi_n^*\rangle.\end{aligned}$$

Since Φ_n^* is a polynomial of degree $\leq n$, it is orthogonal to Φ_{n+1}. Therefore

$$||z\Phi_n||^2 = ||\Phi_{n+1}||^2 + |\alpha_n|^2||\Phi_n^*||^2,$$

which gives us (3.5) because $||z\Phi_n||^2 = ||\Phi_n||^2 = ||\Phi_n^*||^2$. Finally, because

$||\Phi_j|| > 0$ for any j, $|\alpha_n| < 1$ (so $\alpha_n \in \mathbb{D}$). (3.14) follows from induction on (3.5). □

The Szegö recurrence (3.3) and (3.4) can be rearranged into matrix form:

$$\begin{pmatrix} \Phi_{n+1} \\ \Phi^*_{n+1} \end{pmatrix} = \begin{pmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{pmatrix} \begin{pmatrix} \Phi_n \\ \Phi^*_n \end{pmatrix} \tag{3.7}$$

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi^*_{n+1} \end{pmatrix} = A_n(z) \begin{pmatrix} \varphi_n \\ \varphi^*_n \end{pmatrix}, \tag{3.8}$$

where

$$A_n = \rho_n^{-1} \begin{pmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{pmatrix}. \tag{3.9}$$

Note that $A_n(z)$ has the *J-invariance* or *J-unitarity* property. That is, $A_n^* J A_n = J$, for $|z| = 1$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Furthermore,

$$\begin{aligned} A_n^* J A_n &= \rho_n^{-1} \begin{pmatrix} \overline{z} & -\overline{\alpha_n z} \\ -\alpha_n & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rho_n^{-1} \begin{pmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{pmatrix} \\ &= \rho_n^{-2} \begin{pmatrix} \overline{z} & -\overline{\alpha_n z} \\ -\alpha_n & 1 \end{pmatrix} \begin{pmatrix} z & -\overline{\alpha_n} \\ \alpha_n z & -1 \end{pmatrix} \\ &= \rho_n^{-2} \begin{pmatrix} 1 - |\alpha_n|^2 & -\overline{z \alpha_n} - (-\overline{\alpha_n z}) \\ -\alpha_n z + \alpha_n z & -1(1 - |\alpha_n|^2) \end{pmatrix} \\ &= \frac{1}{1 - |\alpha_n|^2} \begin{pmatrix} 1 - |\alpha_n|^2 & 0 \\ 0 & -1(1 - |\alpha_n|^2) \end{pmatrix} = J. \end{aligned}$$

By writing (3.3) and (3.4) in matrix form, we have the following inverse theorem.

Theorem 3.4. *(Inverse Szegö Recurrence) For the monic polynomials*

$$z\Phi_n = \rho_n^{-2}[\Phi_{n+1} + \overline{\alpha_n}\Phi^*_{n+1}] \tag{3.10}$$

$$\Phi^*_n = \rho_n^{-2}[\Phi^*_{n+1} + \alpha_n \Phi_{n+1}], \tag{3.11}$$

and for the orthonormal polynomials

$$\varphi_{n+1} = \rho_n z \varphi_n - \overline{\alpha_n} \varphi^*_{n+1} \tag{3.12}$$
$$\varphi^*_{n+1} = \rho_n \varphi^*_n - \alpha_n \varphi_{n+1}. \tag{3.13}$$

Proof. The proof follows from finding the inverse matrix for the matrix form of (3.3) and (3.4). Namely,

$$\begin{pmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{pmatrix}^{-1} = \frac{1}{z\rho_n^2} \begin{pmatrix} 1 & \overline{\alpha_n} \\ \alpha_n z & z \end{pmatrix}$$

in the matrix form of the Szegö recurrence. □

Next, we prove some bounds for the monic OP's and Verblunsky coefficients.

Theorem 3.5. *For $z \in \partial\mathbb{D}$, we have*

$$\prod_{j=0}^{n-1}(1 - |\alpha_j|) \le \Phi_n(z) \le \prod_{j=0}^{n-1}(1 + |\alpha_j|). \tag{3.14}$$

For $z \in \mathbb{D}$, we have

$$\Phi_n(z) \le \exp\left(\sum_{j=0}^{n-1} |\alpha_j|\right). \tag{3.15}$$

Furthermore, if $B = \sup_j |\alpha_j| < 1$, then

$$\Phi_n(z) \ge \exp\left(-\sum_{j=0}^{n-1} |\alpha_j| - A \sum_{j=0}^{n-1} |\alpha_j|^2\right) \tag{3.16}$$

where $A = \frac{1}{2}(1-B)^{-2}$.

Proof. By the first Szegö Recurrence and the triangle inequality, we have

$$\begin{aligned} |\Phi_{n+1}| &= |z\Phi_n - \overline{\alpha_n}\Phi^*_n| \\ &\le |z\Phi_n| + |\overline{\alpha_n}\Phi^*_n| \\ &= |\Phi_n|(1 + |\alpha_n|) \end{aligned}$$

and

$$\begin{aligned} |\Phi_{n+1}| &= |z\Phi_n - \overline{\alpha_n}\Phi_n^*| \\ &\geq |z\Phi_n| - |\overline{\alpha_n}\Phi_n^*| \\ &= |\Phi_n|(1-|\alpha_n|), \end{aligned}$$

which, with induction, proves (3.14). The rest follows from applying Taylor's remainder theorem to $f(x) = \log(1-x)$. □

Evaluating the first Szegö recurrence (3.3) at $z = 0$, we have

$$\overline{\Phi_{n+1}(0)} = -\alpha_n.$$

This motivates the definition $\alpha_{-1} = -1$. Furthermore, since the quantity $\sqrt{1-|\alpha_j|^2}$ will appear often, we define

$$\rho_j \equiv \sqrt{1-|\alpha_j|^2}.$$

Next, because $||\Phi_n||^2 = \prod_{j=0}^{n}(1-|\alpha_j|^2) = \kappa_n^{-2}$, we have

$$\kappa_n = \prod_{j=0}^{n} \rho_j^{-1}.$$

Since $\varphi_n = \kappa_n\Phi_n$ and $\kappa_{n+1}\rho_n = \kappa_n$, we may recast the Szegö recurrence for the monic polynomials into one for the orthonormal polynomials

$$\frac{\varphi_{n+1}}{\kappa_{n+1}} = z\frac{\varphi_n}{\kappa_n} - \overline{\alpha_n}\frac{\varphi_n^*}{\kappa_n}$$
$$\frac{\varphi_{n+1}^*}{\kappa_{n+1}} = \frac{\varphi_n^*}{\kappa_n} - \alpha_n z\frac{\varphi_n}{\kappa_n},$$

which after rearrangement, gives us the Szegö recurrence for the orthonormal polynomials

$$\varphi_{n+1} = \rho_n^{-1}(z\varphi_n - \overline{\alpha_n}\varphi_n^*) \tag{3.17}$$

$$\varphi_{n+1}^* = \rho_n^{-1}(\varphi_n^* - \alpha_n z\varphi_n). \tag{3.18}$$

Note that after rearranging (3.17) as

$$z\varphi_n = \rho_n \varphi_{n+1} + \overline{\alpha_n}\varphi_n^* \tag{3.19}$$

we have

$$\rho_n^2 + |\alpha_n|^2 = 1.$$

From the proof of Theorem 3.5, we have

$$(1 - |\alpha_n|)|\Phi_n| \le |\Phi_{n+1}| \le (1 + |\alpha_n|)|\Phi_n|,$$

so, using $\rho_n = \frac{\kappa_n}{\kappa_{n+1}}$ and $\Phi_n = \frac{\varphi_n}{\kappa_n}$,

$$(1 - |\alpha_n|)\frac{|\varphi_n|}{\kappa_n} \le |\Phi_{n+1}| \le (1 + |\alpha_n|)\frac{|\varphi_n|}{\kappa_n}$$
$$(1 - |\alpha_n|)|\varphi_n| \le \rho_n|\varphi_{n+1}| \le (1 + |\alpha_n|)|\varphi_n|.$$

Furthermore, note that $\rho_n = \sqrt{1 - |\alpha_n|^2} = \sqrt{(1 + |\alpha_n|)(1 - |\alpha_n|)}$, so the equation above becomes

$$\sqrt{\frac{1 - |\alpha_n|}{1 + |\alpha_n|}}|\varphi_n| \le |\varphi_{n+1}| \le \sqrt{\frac{1 + |\alpha_n|}{1 - |\alpha_n|}}|\varphi_n|, \tag{3.20}$$

which with induction, gives

$$\prod_{j=0}^{n-1}\sqrt{\frac{1 - |\alpha_j|}{1 + |\alpha_j|}} \le |\varphi_n| \le \prod_{j=0}^{n-1}\sqrt{\frac{1 + |\alpha_j|}{1 - |\alpha_j|}}. \tag{3.21}$$

Since $|\alpha_n| < 1$, (3.20) implies the following bounds

$$\sqrt{\frac{1 + |\alpha_n|}{1 - |\alpha_n|}} = \frac{1 + |\alpha_n|}{\rho_n} \le 2\rho_n^{-1}$$
$$\sqrt{\frac{1 - |\alpha_n|}{1 + |\alpha_n|}} = \frac{\rho_n}{\sqrt{1 + |\alpha_n|}} \le \frac{1}{2}\rho_n,$$

which give us, for $z \in \partial\mathbb{D}$:

$$\frac{1}{2}\rho_n|\varphi_n| \leq |\varphi_{n+1}| \leq 2\rho_n^{-1}|\varphi_n|. \tag{3.22}$$

The following theorem shows how the moments c_n are related to the Verblunsky coefficients and the monic OP's. We start with a few definitions. As before, let $\{c_n\}_{n=1}^{\infty}$ be the moments of μ given by

$$c_k = \int_{\partial\mathbb{D}} \exp(-ik\theta) d\mu(\theta) = \langle z^k, 1\rangle.$$

The negative moments of μ are complex conjugates of the positive moments, because

$$\begin{aligned} c_{-k} &= \int_{\partial\mathbb{D}} \exp(ik\theta) d\mu(\theta) \\ &= \overline{\int_{\partial\mathbb{D}} \exp(-ik\theta) d\mu(\theta)} \\ &= \overline{c_k}. \end{aligned}$$

Definition 3.6. (Toeplitz Matrix) Let $\{c_n\}$ be the set of positive and negative moments for μ. The **Toeplitz Matrix** is the following $(n+1)\times(n+1)$ matrix

$$T^{(n+1)} = \begin{pmatrix} c_0 & c_1 & \dots & c_n \\ c_{-1} & c_0 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots \\ c_{-n} & \dots & \dots & c_0 \end{pmatrix}.$$

Definition 3.7. (Toeplitz Determinant) The **Toeplitz Determinant**, $D_n(d\mu)$, is the determinant of $T^{(n+1)}$.

The *Carathédory-Toeplitz theorem* states that $\{c_n\}$ are the moments of a non-trivial measure on $\partial\mathbb{D}$ if and only if $D_n > 0$ for all n (see Simon (2005*b*) for proof).

Theorem 3.8. *(Verblunsky's Formula) Let $\{c_n\}_{n=1}^{\infty}$ be the moments of a probability measure μ and $\{\alpha_j\}_{j=0}^{\infty}$ be its associated Verblunsky coefficients. Then:*

i. The coefficients of Φ_n are polynomials in $\{\alpha_j\}_{j=0}^{n-1}$ and $\{\overline{\alpha_j}\}_{j=0}^{n-1}$ with integer *coefficients. That is:* $\Phi_n(z) = z^n + \sum_{j=0}^{n-1} q_{n,j}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}})z^j$, *where* $q_{n,j}$ *is a polynomial with integer coefficients.*

ii. *For every* n*, there exists a polynomial* $V^{(n)}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}})$ *with integer coefficients so that*

$$c_{n+1} = \alpha_n \prod_{j=0}^{n-1}(1 - |\alpha_j|^2) + V^{(n)}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}}). \quad (3.23)$$

iii. α_n *is a rational function of* $\{c_j\}_{j=1}^{n+1}$ *and* $\{\overline{c_j}\}_{j=0}^{n}$ *where the denominator is the Toeplitz determinant* $D_{n+1}(c)$.

Proof. We prove (i) by using induction on (3.3). The base case is

$$\begin{aligned} \Phi_1(z) &= z \cdot 1 - \overline{\alpha_0} \cdot 1 \\ &= z^1 - (\overline{\alpha_0}) \cdot 1. \end{aligned}$$

The base case is true, as $q_{1,0} = \overline{\alpha_0}$. Now, assuming true for n, we have

$$\begin{aligned} \Phi_{n+1} &= z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \\ &= z(\sum q_{n,j} z^j) - \overline{\alpha_n}(\sum \overline{q_{n,j}} z^{n-j}) \\ &= \sum q_{n,j} z^{j+1} - \sum (\overline{\alpha_n q_{n,j}}) z^{n-j}. \end{aligned}$$

Because the set of polynomials with integer coefficients is closed under addition and multiplication, Φ_{n+1} also has integer coefficients, proving (i). To prove (ii), observe that for each n,

$$\begin{aligned} \int e^{i\theta}\Phi_n d\mu &= \int e^{i\theta}(\frac{\Phi_{n+1} + \overline{\alpha_n}\Phi_n^*}{z})d\mu \\ &= \int \Phi_n d\mu + \overline{\alpha_n}\int \Phi_n^* d\mu \\ &= \langle 1, \Phi_{n+1}\rangle + \overline{\alpha_n}\langle 1, \Phi_n^*\rangle \\ &= \overline{\alpha_n}||\Phi_n||^2 \\ &= \overline{\alpha_n}\prod_{j=0}^{n-1} 1 - |\alpha_j|^2. \end{aligned}$$

From (i), we also have

$$\begin{aligned}
\int e^{i\theta}\Phi_n d\mu &= \int e^{i\theta}\left(z^n + \sum_{j=0}^{n-1} q_{n,j} z^j\right) d\mu \\
&= \int e^{i\theta} z^n d\mu + \sum_{j=0}^{n-1} e^{i\theta} z^j q_{n,j} \\
&= \overline{c_{n+1}} + \sum_{j=0}^{n-1}\left(\int z^{j+1} d\mu\right) q_{n,j} \\
&= \overline{c_{n+1}} + \sum_{j=0}^{n-1} \overline{c_{j+1}} q_{n,j}.
\end{aligned}$$

Thus

$$\overline{\alpha_n}\prod_{j=0}^{n-1}(1-|\alpha_j|^2) = \overline{c_{n+1}} + \sum_{j=0}^{n-1}\overline{c_{j+1}} q_{n,j}.$$

The proof of (ii) follows from taking the complex conjugate of the equation above and using induction. The proof of (iii) is Heine's formula in (3.35). □

The following theorem allows us to compare several notions of convergence for the various objects and parameters related to OPUC. For the proof, the reader is directed to Simon (2005*b*).

Theorem 3.9. *Let $\{d\mu_m\}$ and $d\mu_\infty$ be probability measures on $\partial\mathbb{D}$. Let $c_j(d\mu_m)$, $\alpha_j(d\mu_m)$, $F(z, d\mu_m)$, $f(z, d\mu_m)$, $\Phi_j(z; d\mu_m)$, and $\varphi_j(z; d\mu_m)$ be their moments, Verblunsky coefficients, m-functions, Schur functions, monic orthogonal polynomials, and orthonormal polynomials. The following are equivalent:*

i. $d\mu_m \to d\mu_\infty$ *weakly. That is,* $\int f d\mu_m \to \int f d\mu$ *for all bounded, continuous functions* f*.*

ii. $c_j(d\mu_m) \to c_j(d\mu_\infty)$ *for each* j*,*

iii. $\alpha_j(d\mu_m) \to \alpha_j(d\mu_\infty)$ *for each* j*,*

iv. $F(z, d\mu_m) \to F(z, d\mu_\infty)$ *uniformly on each compact subset of* $\mathbb{D}$*,*

v. $f(z, d\mu_m) \to f(z, d\mu_\infty)$ *uniformly on each compact subset of* $\mathbb{D}$*,* *vi.* $\Phi_j(z; d\mu_m) \to \Phi_j(z, d\mu_\infty)$ *for each* j*, and*

vii. $\varphi_j(z; d\mu_m) \to \varphi_j(z, d\mu_\infty)$ *for each* j*.*

Theorem 3.10. *With a non-trivial probability measure* μ *on* $\partial\mathbb{D}$*, the following are equivalent:*

i. $\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty$,

ii. $\lim_{n\to\infty} \kappa_n < \infty$*, and*

iii. The polynomials z *are not dense in* $L^2(\partial\mathbb{D}, \mathrm{d}\mu)$*.*

Remark 3.11. Because $||\Phi_{n+1}||^2 = (1 - |\alpha_n|^2)||\Phi_n||^2$ and $|\alpha_n| < 1$, we have for all n, $||\Phi_{n+1}|| \leq ||\Phi_n||$ and $\kappa_{n+1} \geq \kappa_n$.

Proof. To see that (i) and (ii) are equivalent, observe the following. For a sequence of complex numbers $\{\alpha_j\}_{j=1}^{\infty}$, we have the following chain of inequalities:

$$
\begin{aligned}
\prod_{j=0}^{\infty}(1 - |\alpha_j|^2) &> 0 \\
\log\left(\prod_{j=0}^{\infty}(1 - |\alpha_j|^2)\right) &> -\infty \\
\sum_{j=0}^{\infty}\log(1 - |\alpha_j|^2) &> -\infty \\
\sum_{j=0}^{\infty} -|\alpha_j|^2 &> -\infty \\
\sum_{j=0}^{\infty} |\alpha_j|^2 &< \infty.
\end{aligned}
$$

Therefore, (i) is equivalent to $\prod_{j=0}^{\infty}(1 - |\alpha_j|^2) > 0$. Because $\prod_{j=0}^{n-1}(1 - |\alpha_j|^2) = \kappa_n^{-2}$,

$$
\begin{aligned}
\prod_{j=0}^{\infty}(1 - |\alpha_j|^2) &> 0 \\
\lim_{n\to\infty} \kappa_n^{-2} &> 0 \\
\lim_{n\to\infty} \kappa_n &< \infty
\end{aligned}
$$

which shows that (i) is equivalent to (ii). Next, we show that (ii) implies (iii). As Φ_n is the projection of z^n onto the orthogonal complement of $\text{span}(1, z, \ldots, z^{n-1})$, by the anti-unitarity of R_n, $\Phi_n^* = R_n\Phi_n$ is the projection of $R_n(z^n) = 1$ onto the orthogonal complement of $\text{span}(R_n(1), R_n(z), \ldots, R_n(z^{n-1})) = \text{span}(z, z^2, \ldots, z^n)$. Furthermore, multiplication by z^{-1} is unitary, as

$$\begin{aligned} \langle \overline{z}f, \overline{z}g \rangle &= \int \overline{\overline{z}f}\overline{z}g \quad d\mu \\ &= \int (fg) \quad d\mu \\ &= \langle f, g \rangle. \end{aligned}$$

So, $z^{-1}\Phi_n^*$ is the projection of z^{-1} onto the orthogonal complement of the span of $1, z, \ldots, z^{n-1}$. Now, let Q be the projection onto the closure of $\text{span}(1, z, z^2, \ldots)$. It follows that $1 - Q$ is the projection onto the orthogonal complement of the closure of $\text{span}(1, z, z^2, \ldots)$. Taking the norm,

$$\begin{aligned} ||(1 - Q)z^{-1}|| &= \lim_{n\to\infty} ||z^{-1}\Phi_n^*|| \\ &= \lim_{n\to\infty} (||z^{-1}||)(||\Phi_n^*||) \\ &= \lim_{n\to\infty} ||\Phi_n|| \\ &= \lim_{n\to\infty} \kappa_n^{-1}. \end{aligned}$$

Therefore, (ii) implies that, for all $z \in \partial\mathbb{D}$:

$$\begin{aligned} ||(1 - Q)z^{-1}|| &\neq 0 \\ (1 - Q)z^{-1} &\neq 0 \\ z^{-1} - Qz^{-1} &\neq 0, \end{aligned}$$

so that the projection of z^{-1} onto the closure of $\text{span}(z, z^2, \ldots)$ is non-zero, implying that the polynomials are not dense in $L^2(\partial\mathbb{D}, \mathrm{d}\mu)$.

Finally, to see that (iii) implies (ii), we prove the converse that if $\lim_{n\to\infty} \kappa_n = \infty$, then the polynomials are dense. If $\lim_{n\to\infty} \kappa_n = \infty$, then by the previous part, $||(1 - Q)z^{-1}|| = 0$ and hence $(1 - Q)z^{-1} = 0$. In that case, we can prove by induction that $(1-Q)z^{-l} = 0$ for all $l \geq 1$. This implies that the polynomials z are dense in the Laurent polynomials, which shows that the polynomials are dense in L^2 because the Laurent polynomials are dense in L^2. □

Next, we want to be able to express inner products of the form $\langle f, z^l g\rangle$, where f, g are the orthonormal or the reverse orthonormal polynomials. To do so, we first need the following lemma.

Lemma 3.12. *The following identities hold:*

i.

$$\langle \varphi_j, \varphi_k\rangle = \delta_{jk}.$$

ii.

$$\langle \varphi_j^*, \varphi_k\rangle = \begin{cases} -\overline{\alpha_{k-1}}\prod_{l=k}^{j-1}\rho_l & \text{if } 0 \le k \le j \\ 0 & \text{if } k \ge j+1. \end{cases}$$

iii.

$$\langle \varphi_j, \varphi_k^*\rangle = \begin{cases} -\alpha_{j-1}\prod_{l=k}^{k-1}\rho_l & \text{if } 0 \le j \le k \\ 0 & \text{if } j \ge jk+1. \end{cases}$$

iv.

$$\langle \varphi_j^*, \varphi_k^*\rangle = \prod_{l=\min(k,j)}^{\max(k,j)-1}\rho_l. \tag{3.24}$$

Remark 3.13. (a). The inner product for the monic polynomials can be easily deduced from the relations above because they only differ by a multiplicative constant κ_n.

(b). The corner case $\prod_{l=j}^{j-1}\rho_l$ is interpreted as 1.

Proof. (i) is trivial from the Gram-Schmidt construction. Note that (ii) is the complex conjugate form of (iii), so it suffices to prove (ii) and (iv). To prove (ii), note that we now Φ_j^* is orthogonal to $z, \dots, z^j$. Therefore, If P is any polynomial of degree $\leq j$, then

$$\begin{aligned}\langle \Phi_j^*, P - P(0)\rangle &= 0\\ \langle \Phi_j^*, P\rangle &= \langle \Phi_j^*, P(0)\rangle\\ &= P(0)\langle \Phi_j^*, 1\rangle\\ &= P(0)||\Phi_j^*||^2.\end{aligned}$$

Thus, for $k \leq j$, we have

$$\begin{aligned}\langle \Phi_j^*, \Phi_k\rangle &= \Phi_k(0)||\Phi_j^*||^2\\ \langle \varphi_j^*, \varphi_k\rangle &= \frac{||\Phi_j||}{||\Phi_k||}\Phi_k(0).\end{aligned}$$

Because $\alpha_{n-1} = \overline{\Phi_n(0)}$,

$$\begin{aligned}\langle \varphi_j^*, \varphi_k\rangle &= \frac{||\Phi_j||}{||\Phi_k||}(-\overline{\alpha_{k-1}})\\ &= \frac{\prod_{l=0}^{j-1}(1-|\alpha_l|^2)}{\prod_{l=0}^{k-1}(1-|\alpha_l|^2)}(-\overline{\alpha_{k-1}})\\ &= -\overline{\alpha_{k-1}}\prod_{l=k}^{j-1}\rho_l,\end{aligned}$$

which proves (ii). To get (iv), suppose without loss of generality that $1 \leq k \leq j$. Then

$$\begin{aligned}\langle \varphi_j^*, \varphi_k^* - \varphi_k^*(0)\rangle &= 0\\ \langle \varphi_j^*, \varphi_k^*\rangle &= \langle \varphi_j^*, \varphi_k^*(0)\rangle\\ &= \varphi_k^*(0)\langle \varphi_j^*, 1\rangle\\ &= \frac{\varphi_k^*(0)}{||\Phi_j||}\langle ||\Phi_j||\varphi_j^*, 1\rangle\\ &= \varphi_k^*(0)||\Phi_j||\\ &= \Phi_k^*(0)\frac{||\Phi_j||}{||\Phi_k||},\end{aligned}$$

since $\Phi_k^*(0) = 1$ and

$$\begin{aligned}\langle \varphi_j^*, \varphi_k^* \rangle &= \frac{||\Phi_j||}{||\Phi_k||} \\ &= \frac{\prod_{l=0}^{l=j-1} \rho_l}{\prod_{l=0}^{l=k-1} \rho_l},\end{aligned}$$

completing the proof for (iv). □

Finally, we are able to establish the following relations for inner products.

Theorem 3.14. *i.*

$$\langle \varphi_j, z\varphi_k \rangle = \begin{cases} -\overline{\alpha_k}\alpha_{j-1} \prod_{l=j}^{k-1} \rho_l & \textit{if } 0 \leq j \leq k \\ \rho_k & \textit{if } j = k+1 \\ 0 & \textit{if } j \geq k+2. \end{cases}$$

ii.

$$\begin{aligned}
\langle \varphi_{2n+1}, z\varphi_{2n+1} \rangle &= -\overline{\alpha}_{2n+1}\alpha_{2n} && (3.25)\\
\langle \varphi_{2n+3}, z^2\varphi_{2n+1} \rangle &= \rho_{2n+1}\rho_{2n+2} && (3.26)\\
\langle \varphi_{2n}^*, z\varphi_{2n+1} \rangle &= \overline{\alpha}_{2n+1}\rho_{2n} && (3.27)\\
\langle \varphi_{2n+2}^*, z^2\varphi_{2n+1} \rangle &= \overline{\alpha}_{2n+2}\rho_{2n+1} && (3.28)\\
\langle \varphi_{2n}^*, z\varphi_{2n}^* \rangle &= -\overline{\alpha}_{2n}\alpha_{2n-1} && (3.29)\\
\langle \varphi_{2n-2}^*, \varphi_{2n}^* \rangle &= \rho_{2n-2}\rho_{2n-1} && (3.30)\\
\langle \varphi_{2n+1}, z\varphi_{2n}^* \rangle &= -\alpha_{2n-1}\rho_{2n} && (3.31)\\
\langle \varphi_{2n-1}, \varphi_{2n}^* \rangle &= -\alpha_{2n-2}\rho_{2n-1}. && (3.32)
\end{aligned}$$

Proof. To show (i), we have from the Szegö recurrence

$$z\varphi_n = \rho_n\varphi_{n+1} + \overline{\alpha}_n\varphi_n^*.$$

Thus

$$\langle \varphi_j, z\varphi_k \rangle = \rho_k\langle \varphi_j, \varphi_{k+1} \rangle + \overline{\alpha}_k\langle \varphi_j, \varphi_k^* \rangle,$$

which, by Lemma 3.12, gives us

$$\langle \varphi_j, z\varphi_k \rangle = \begin{cases} 0 & \text{if } 0 \leq j \leq k \\ \rho_k & \text{if } j = k+1 \\ 0 & \text{if } j \geq k+2 \end{cases} + \begin{cases} -\overline{\alpha_k}\alpha_{j-1} \prod_{l=j}^{k-1} \rho_l & \text{if } 0 \leq j \leq k \\ 0 & \text{if } j \geq k+1 \end{cases}$$

$$= \begin{cases} -\overline{\alpha_k}\alpha_{j-1} \prod_{l=j}^{k-1} \rho_l & \text{if } 0 \leq j \leq k \\ \rho_k & \text{if } j = k+1 \\ 0 & \text{if } j \geq k+2 \end{cases}$$

Next we prove (ii). First, note that (3.25) and (3.29) are special cases of (i), because

$$\begin{aligned} \langle \varphi_{2n+1}, z\varphi_{2n+1} \rangle &= -\overline{\alpha}_{2n+1}\alpha_{2n} \prod_{2n+1}^{2n} \rho_l \\ &= -\overline{\alpha}_{2n+1}\alpha_{2n} \\ \langle \varphi_{2n}^*, z\varphi_{2n}^* \rangle &= \langle z^{-2n}\overline{\varphi}_{2n}, z^{-2n+1}\overline{\varphi}_{2n} \rangle \\ &= \int z^{2n}\varphi_2 n z^{-2n+1}\overline{\varphi}_{2n} d\mu \\ &= \int (\overline{\varphi}_{2n}) z\varphi_{2n} d\mu \\ &= \langle \varphi_{2n}, z\varphi_{2n} \rangle \\ &= -\overline{\alpha}_{2n}\alpha_{2n-1} \end{aligned}$$

Equations (3.30) and (3.32) are special cases of the previous theorem:

$$\begin{aligned} \langle \varphi_{2n-2}^*, \varphi_{2n}^* \rangle &= \prod_{\min(2n-2,2n)}^{\max(2n-2,2n)-1} \rho_l \\ &= \rho_{2n-2}\rho_{2n-1} \\ \langle \varphi_{2n-1}^*, \varphi_{2n}^* \rangle &= -\alpha_{2n-2} \prod_{2n-1}^{2n-1} \rho_l \\ &= -\alpha_{2n-2}\rho_{2n-1}. \end{aligned}$$

Equations (3.26) and (3.27) are also special cases of lemma 3.12.

$$\begin{aligned}
\langle \varphi_{2n+3}, z^2\varphi_{2n+1}\rangle &= \langle z^{-2}\overline{\varphi}_{2n+1}, \overline{\varphi}_{2n+3}\rangle \\
&= \langle z^{2n+3}z^{-2}\overline{\varphi}_{2n+1}, z^{2n+3}\overline{\varphi}_{2n+3}\rangle \\
&= \langle \varphi^*_{2n+1}, \varphi^*_{2n+3}\rangle \\
&= \rho_{2n+1}\rho_{2n+2} \\
\langle \varphi^*_{2n}, z\varphi_{2n+1}\rangle &= \langle \varphi^*_{2n}, \rho_{2n+1}\varphi_{2n+2} + \overline{\alpha}_{2n+1}\varphi^*_{2n+1}\rangle \\
&= 0 + \overline{\alpha}\langle \varphi^*_{2n}, \varphi^*_{2n+1}\rangle \\
&= \overline{\alpha}_{2n+1}\rho_{2n}.
\end{aligned}$$

To obtain (3.28), we first multiply z to both sides of the Szegö recurrence

$$z^2\varphi_{2n+1} = \rho_{2n+1}z\varphi_{2n+2} + \overline{\alpha}_{2n+1}z\varphi^*_{2n+1}$$

so that

$$\begin{aligned}
\langle \varphi^*_{2n+2}, z^2\varphi_{2n+1}\rangle &= \langle \varphi^*_{2n+2}, \rho_{2n+1}z\varphi_{2n+2}\rangle + \langle \varphi^*_{2n+2}, \overline{\alpha}_{2n+1}z\varphi^*_{2n+1}\rangle \\
&= \langle \varphi^*_{2n+2}, \rho_{2n+1}(\rho_{2n+2}\varphi_{2n+3} + \overline{\alpha}_{2n+2}\varphi^*_{2n+2})\rangle + \overline{\alpha}_{2n+1}\langle \varphi^*_{2n+2}, z\varphi^*_{2n+1}\rangle.
\end{aligned}$$

As φ_{2n+3} and φ^*_{2n+2} are orthogonal and $z\varphi^*_{2n+1}$ is of degree ≥ 1 and $\leq 2n+2$, it is orthogonal to φ^*_{2n+2}. We thus have

$$\begin{aligned}
\langle \varphi^*_{2n+2}, z^2\varphi_{2n+1}\rangle &= \langle \varphi^*_{2n+2}, \overline{\alpha}_{2n+2}\varphi^* 2n+2\rangle \\
&= \overline{\alpha}_{2n+2}\rho_{2n+1}.
\end{aligned}$$

Finally, to get (3.31), note

$$\begin{aligned}
\langle \varphi_{2n-1}, \varphi^*_{2n}\rangle &= \langle z^{-1}\overline{\varphi}^*_{2n}, \overline{\varphi}_{2n+1}\rangle \\
&= \langle z^{2n+1}z^{-1}\overline{\varphi^*}_{2n}, z^{2n+1}\overline{\varphi}_{2n+1}\rangle \\
&= \langle \varphi_{2n}, \varphi^*_{2n+1}\rangle \\
&= -\overline{\alpha}_{2n-1}\rho_{2n}.
\end{aligned}$$

□

Proposition 3.15. *We have*

$$\langle \varphi_j^*, z\varphi_k \rangle \quad = \quad \begin{cases} \overline{\alpha}_k \prod_{l=k}^{j-1} \rho_l & \text{if } j \leq k \\ 0 & \text{if } j \geq k+1. \end{cases} \tag{3.33}$$

Moreover,

$$\overline{\alpha}_k \quad = \quad \langle \varphi_k^*, z\varphi_k \rangle = \int_{\partial\mathbb{D}} (\varphi_k)^2 z^{1-k} d\mu. \tag{3.34}$$

Proof. For $j \geq k+1$, since $1 \leq \deg(z\varphi_k) \leq j$, by the property of φ_j^*, $\langle \varphi_j^*, z\varphi_k \rangle = 0$. For $j \leq k$, we can use (3.19), the orthonormal version of the inverse Szegö recurrence:

$$\begin{aligned} \langle \varphi_j^*, z\varphi_k \rangle &= \langle \varphi_j^*, \rho_k \varphi_{k+1} + \overline{\alpha}_k \varphi_k^* \rangle \\ &= \langle \varphi_j^*, \overline{\alpha}_k \varphi_k^* \rangle \\ &= \overline{\alpha}_k \prod_{l=j}^{k-1} \rho_l, \end{aligned}$$

where the last line follows from (3.24). To see (3.34):

$$\begin{aligned} \overline{\alpha}_k &= \langle \varphi_k^*, z\varphi_k \rangle \\ &= \int \overline{\varphi_k^*} z \varphi_k d\mu \\ &= \int (z^k \overline{\varphi_k^*}) z^{1-k} \varphi_k d\mu \\ &= \int z^{1-k} (\varphi_k)^2 d\mu. \end{aligned}$$

□

The following theorem relates the orthogonal polynomials Φ_n with the Toeplitz determinants D_n and the Verblunsky coefficients α_n.

Theorem 3.16. *(a) (Heine's Formula) Let $\{c_n\}$ be the moments of $d\mu$ and*

$\{\Phi_n\}$ *its orthogonal polynomials. Then*

$$\Phi_n(z) = D_{n-1}(d\mu)^{-1} \begin{vmatrix} c_0 & \bar{c}_1 & \dots & \bar{c}_n \\ c_1 & c_0 & \dots & \bar{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \dots & \bar{c}_1 \\ 1 & z & \dots & z^n \end{vmatrix}, \tag{3.35}$$

(b)

$$||\Phi_n(z)||^2 = \prod_{j=0}^{n-1}(1-|\alpha_j|^2) = \frac{D_n(d\mu)}{D_{n-1}(d\mu)}, \tag{3.36}$$

(c)

$$1-|\alpha_n|^2 = \frac{D_{n+1}D_{n-1}}{D_n^2}, \tag{3.37}$$

(d)

$$\alpha_n = (-1)^n \frac{D_n(e^{-i\theta}d\mu)}{D_n(d\mu)}. \tag{3.38}$$

Remark 3.17. (1) (b) is a recursive formula that allows us to compute the Toeplitz determinants from the Verblunsky coefficients.

(2) (b) shows that the Toeplitz determinants are real.

(3) (d) allows us to solve for $c_1, c_2, \dots, c_n$ recursively from $\alpha_0, \alpha_1, \dots, \alpha_n$. This is the explicit formula for $V^{(n)}$ in Theorem 3.8 in determinant form.

Proof. The RHS of (3.35) is clearly a polynomial of degree $\leq n$, which we denote as $P(z)$. To show that $P(z) = \Phi_n(z)$, it is enough to show that it is monic and that it is orthogonal to $1, z, \dots, z^{n-1}$. Rearranging (3.35), we have

$$D_{n-1}(d\mu)P(z) = \begin{vmatrix} c_0 & \overline{c}_1 & \dots & \overline{c}_n \\ c_1 & c_0 & \dots & \overline{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \dots & \overline{c}_1 \\ 1 & z & \dots & z^n \end{vmatrix}.$$

For $j = 0, 1, \dots, n-1$,

$$\begin{aligned} D_{n-1}(d\mu)\langle z^j, P(z)\rangle &= \langle z^j, \begin{vmatrix} c_0 & \overline{c}_1 & \dots & \overline{c}_n \\ c_1 & c_0 & \dots & \overline{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \dots & \overline{c}_1 \\ 1 & z & \dots & z^n \end{vmatrix} \rangle \\ &= \int z^{-j} \begin{vmatrix} c_0 & \overline{c}_1 & \dots & \overline{c}_n \\ c_1 & c_0 & \dots & \overline{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \dots & \overline{c}_1 \\ 1 & z & \dots & z^n \end{vmatrix} d\mu. \end{aligned}$$

Expanding the determinant along the last row, taking integrals to get the moments, and translating the sum back into a determinant, we get

$$D_{n-1}(d\mu)\langle z^j, P(z)\rangle = \begin{vmatrix} c_0 & \overline{c}_1 & \dots & \overline{c}_n \\ c_1 & c_0 & \dots & \overline{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \dots & \overline{c}_1 \\ c_j & c_{j-1} & \dots & c_{j-n} \end{vmatrix} d\mu.$$

Note that for $j = 0, 1, \dots, n-1$, the last row of the matrix above is identical to the $(j+1)$-th row, so the matrix is singular and the determinant is zero. Because $D_{n-1}(d\mu)$ is non-zero, it follows that $P(z)$ must be orthogonal to $1, z, \dots, z^{n-1}$. Next, to see that $P(z)$ is monic, expand the RHS of (3.35) along the last row, obtaining

$$\begin{aligned}
P(z) &= z^n \begin{vmatrix} c_0 & \overline{c}_1 & \dots & \overline{c}_{n-1} \\ c_1 & c_0 & \dots & \overline{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & \dots & \overline{c}_0 \end{vmatrix} D_{n-1}(d\mu)^{-1} + \text{lower order terms} \\
&= z^n \det(T^{(n)}) D_{n-1}(d\mu)^{-1} + \text{lower order terms} \\
&= z^n + \text{lower order terms.}
\end{aligned}$$

This proves (3.35). To show (3.36), let $j = n$ in the steps above. Using the fact that $\det(A^*) = \overline{\det(A)}$ and that the Toeplitz determinant is real, we see that

$$\langle z^n, \Phi_n(z) \rangle = \frac{D_n(d\mu)}{D_{n-1}(d\mu)}.$$

We also know $||\Phi_n(z)||^2 = \prod_{j=0}^{n-1}(1 - |\alpha_j|^2)$ from (3.6), which proves (b). To show (c), simply note that

$$\left(\frac{D_{n+1}(d\mu)}{D_n(d\mu)} \right) \left(\frac{D_{n-1}(d\mu)}{D_n(d\mu)} \right) = \prod_{j=0}^{n} \rho_j \prod_{j=0}^{n-1} \rho_j^{-1} = \rho_n = (1 - |\alpha_n|^2).$$

Finally, to get (d), evaluate (3.35) at $z = 0$ and expand along the last row:

$$\begin{aligned}
D_{n-1}(d\mu)\Phi_n(0) &= (-1)^n \begin{vmatrix} \overline{c}_1 & \overline{c}_2 & \dots & \overline{c}_n \\ c_0 & \overline{c}_1 & \dots & \overline{c}_{n-1} \\ \vdots & \vdots & & \vdots \\ c_{n-2} & \dots & \dots & \overline{c}_1 \end{vmatrix} \\
&= (-1)^n \begin{vmatrix} c_{-1} & \dots & \dots & c_{-n} \\ c_0 & c_{-1} & \dots & c_{-n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-2} & \dots & \dots & c_{-1} \end{vmatrix}.
\end{aligned}$$

Recognizing that if we take the transformation of $d\mu \to e^{i\theta} d\mu$ then $c_j \to c_{j-1}$, we have:

$$D_{n-1}(d\mu)\Phi_n(0) = (-1)^n D_{n-1}(e^{i\theta}d\mu).$$

Using $\alpha_n = -\overline{\Phi_{n+1}(0)}$ and that the Toeplitz determinant is real, we have

$$\begin{aligned} -\overline{\alpha}_{n-1} &= (-1)^n \frac{D_{n-1}(e^{i\theta}d\mu)}{D_{n-1}(d\mu)} \\ \alpha_{n-1} &= (-1)^{n-1} \frac{D_{n-1}(e^{-i\theta}d\mu)}{D_{n-1}(d\mu)}, \end{aligned}$$

which completes the proof of (d). □

The following formula allows us to compute the Toeplitz determinant directly from the measure.

Theorem 3.18.

$$D_n(d\mu) = \frac{1}{(n+1)!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{0\le j<k\le n}|e^{i\theta_j}-e^{i\theta_k}|^2\prod_{j=0}^{n}d\mu(\theta_j). \qquad (3.39)$$

Before the proof, we verify this formula for the case $n = 1$:

$$\begin{aligned} LHS &= D_1(d\mu) \\ &= \det\begin{pmatrix} c_0 & c_1 \\ \overline{c}_1 & c_0 \end{pmatrix} \\ &= \det\begin{pmatrix} 1 & c_1 \\ \overline{c}_1 & 1 \end{pmatrix} \\ &= 1-|c_1|^2, \end{aligned}$$

While

$$
\begin{aligned}
RHS &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} |e^{i\theta_0} - e^{i\theta_1}|^2 d\mu(\theta_0)d\mu(\theta_1) \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} (2 - e^{i(\theta_0-\theta_1)} - e^{i(\theta_1-\theta_0)})d\mu(\theta_0)d\mu(\theta_1) \\
&= \int_0^{2\pi}\int_0^{2\pi} d\mu(\theta_0)d\mu(\theta_1) - \frac{1}{2}\Bigg[\left(\int_0^{2\pi} e^{i\theta_1}d\mu(\theta_1)\right)\left(\int_0^{2\pi} e^{-i\theta_0}d\mu(\theta_0)\right) \\
&+ \left(\int_0^{2\pi} e^{-i\theta_1}d\mu(\theta_1)\right)\left(\int_0^{2\pi} e^{i\theta_0}d\mu(\theta_0)\right)\Bigg] \\
&= 1 - |c_1|^2.
\end{aligned}
$$

Proof. Let Σ_{n+1} be the permutation group of $\{0, 1, \ldots, n\}$. We can rewrite the right hand side as

$$
\begin{aligned}
RHS &= \frac{1}{(n+1)!}\sum_{\pi,\sigma\in\Sigma_{n+1}} (-1)^{\pi}(-1)^{\sigma}\int_0^{2\pi}\ldots\int_0^{2\pi}\prod_{j=0}^{n} e^{i(\sigma(j)-\pi(j))\theta_j}d\mu(\theta_j) \\
&= \frac{1}{(n+1)!}\sum_{\pi,\sigma}(-1)^{\pi\sigma^{-1}}\left(\int_0^{2\pi} e^{i(\sigma(0)-\pi(0))\theta_0}d\mu(\theta_0)\right)\ldots\left(\int_0^{2\pi} e^{i(\sigma(n)-\pi(n))\theta_j}d\mu(\theta_n)\right) \\
&= \frac{1}{(n+1)!}\sum_{\pi,\sigma}(-1)^{\pi\sigma^{-1}}\prod_{j=0}^{n}\overline{c_{\sigma(j)-\pi(j)}} \\
&= \frac{1}{(n+1)!}\sum_{\pi,\sigma}(-1)^{\pi\sigma^{-1}}\prod_{j=0}^{n} T^{(n+1)}(d\mu)_{\pi(j)\sigma(j)}.
\end{aligned}
$$

First summing over π for σ fixed, we get

$$
RHS = \frac{1}{(n+1)!}\sum_{\sigma}\det(T^{(n+1)}(d\mu)),
$$

and then summing over σ, which yields a factor of $(n+1)!$ and completes the proof. □

Theorem 3.19. *Let L_N be the transpose of $T^{(N+1)}$, that is, $(L_N)_{jk} = c_{j-k}$. Let*

$$
\Phi_N(z) = \sum_{k=0}^{N} a_k^{(N)} z^k
$$

and let $\boldsymbol{a}$ *be the column vector with components* $a_j^{(N)}$. *Then* $\boldsymbol{a}$ *(and so* Φ*) is determined uniquely by the equations*

$$\begin{aligned} a_N^{(N)} &= 1 \\ (L_N \boldsymbol{a})_j = &= 0. \end{aligned}$$

Furthermore, the Verblunsky coefficients are given by

$$\overline{\alpha_{N-1}} = -\frac{\langle \delta_0, L_N^{-1} \delta_N \rangle}{\langle \delta_N, L_N^{-1} \delta_N \rangle}.$$

Proof. For the proof, see Simon (2005*b*). □

4 Examples of OPUC

In this section, we present various classes of OPUC by computing explicit formulae for the orthogonal polynomials and the related mathematical objects existing in bijection. The examples in Section 4.1 are well-known and is recorded in Chapter 1.6 of Simon (2005*b*). In Section 4.2, we study several new classes of OPUC and provide a new theorem relating the even Verblunsky coefficients with the odd moments. These examples demonstrate how the various mathematical objects associated to OPUC connect with each other. In the following examples, let $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$ be the Radon-Nikodym decomposition of the measure $d\mu$.

4.1 Classical Examples

Example 4.1. (The Lebesgue Measure) Let $w(\theta) = 1$. We have that $d\mu_s = 0$ because $\int w(\theta)\frac{d\theta}{2\pi} = 1$. Computing the moments, we get

$$c_n = \int e^{-in\theta} d\mu(\theta) = \delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker-delta function. The monic OP's can be directly computed from Gram-Schmidt (alternatively, we could use (3.35) and expand along the last column):

$$\begin{aligned}
\Phi_0 &= 1 \\
\Phi_1 &= z - \langle z, \Phi_0 \rangle \frac{\Phi_0}{||\Phi_0||^2} \\
&= z - \int z^{-1} d\mu = z \\
\Phi_2 &= z^2 - \langle z^2, \Phi_1 \rangle \frac{\Phi_1}{||\Phi_1||^2} - \langle z^2, \Phi_0 \rangle \frac{\Phi_0}{||\Phi_0||^2} \\
&= z^2 - \left(\int z^{-1} d\mu \right) \frac{\Phi_1}{||\Phi_1||^2} - \left(\int z^{-2} d\mu \right) \frac{\Phi_0}{||\Phi_0||^2} \\
&= z^2.
\end{aligned}$$

Therefore, we can inductively show that $\Phi_n = z^n$. It follows that $\Phi_n^* = 1$ for all n. Because z^n is of length 1, the monic polynomials are the same as the orthonormal polynomials. Furthermore, $\alpha_n = -\overline{\Phi_{n+1}(0)} = 0$. The Toeplitz matrix is simply the identity matrix for all $n \geq 1$, so its determinant is $D_n = 1$. Because $c_n = 0$ for $n \geq 1$, the Carathédory function is $F(z) = 1$. It follows that the Schur function is $f(z) = 0$. Finally, the Szegö function $D(z) = (\lim_{n\to\infty} \varphi_n^*)^{-1} = 1$.

Example 4.2. (Bernstein-Szegö Polynomials) Define the moments as $c_n \equiv \gamma^n$ for $n \geq 0$, where $\gamma = re^{i\varphi}$ and $0 < r < 1$. It follows that $c_{-n} = \overline{c_n} = \overline{\gamma}^n$. We claim $\Phi_n = z^n - \overline{\gamma} z^{n-1}$. To see this, for $0 \leq j \leq n-1$:

$$\begin{aligned} \langle z^j, z^n - \overline{\gamma} z^{n-1} \rangle &= \int z^{-j}(z^n - \overline{\gamma} z^{n-1}) d\mu \\ &= \int z^{-(j-n)} - \overline{\gamma} \int z^{-(j-(n-1))} \\ &= \overline{c}_{n-j} - \overline{\gamma c}_{n-j-1} \\ &= \overline{\gamma}^{-(n-j)} - \overline{\gamma}^{-(n-j)} = 0. \end{aligned}$$

Next, $\Phi_n^*(z) = z^n \overline{z^n - \overline{\gamma} z^{n-1}} = z^n(z^{-n} - \gamma z^{1-n}) = 1 - \gamma z$. Because $\alpha_n = -\overline{\Phi_{n+1}(0)}$, $\alpha_0 = \gamma$ and $\alpha_j = 0$ for $j \geq 1$. From (3.6), $||\Phi_n||^2 = ||\Phi_n^*||^2 = \prod_{j=0}^{n-1}(1 - |\alpha_j|^2)$. In this case, we obtain $||\Phi_n||^2 = 1 - |\gamma|^2$ for $n \geq 1$, so

$$\begin{aligned} \varphi_n &= (1 - |\gamma|^2)^{-\frac{1}{2}}(z^n - \overline{\gamma} z^{n-1}) \\ \varphi_n^* &= (1 - |\gamma|^2)^{-\frac{1}{2}}(1 - \gamma z). \end{aligned}$$

The Carathédory function $F(z)$ is a geometric series because the moments are simple powers of n:

$$\begin{aligned} F(z) &= 1 + 2\sum_{n=1}^{\infty} c_n z^n \\ &= 1 + 2\sum_{n=1}^{\infty} (\gamma z)^n = \frac{1 + \gamma z}{1 - \gamma z}, \end{aligned}$$

which implies that $f(z) = \gamma$. Next, to obtain the density $w(\theta)$, we take the real part of the Carathédory function:

$$\begin{aligned} w(\theta) &= \text{Re } F(e^{i\theta}) \\ &= \frac{1}{2}\left(\frac{1+\gamma z}{1-\gamma z} + \frac{1+\overline{\gamma z}}{1-\overline{\gamma z}}\right) \\ &= \frac{1-|\gamma|^2}{|1-\gamma z|^2}; \end{aligned}$$

note $d\mu_s = 0$ because the a.c. part integrates to 1. We recursively find that $D_n = (1-|\gamma|^2)^n$. Finally,

$$\begin{aligned} D(z)^{-1} &= \lim_{n\to\infty} \varphi_n^*(z) \\ &= \lim_{n\to\infty} (1-|\gamma|^2)^{-\frac{1}{2}}(1-\gamma z) \\ &= (1-|\gamma|^2)^{-\frac{1}{2}}(1-\gamma z). \end{aligned}$$

Example 4.3. (Single Inserted Mass Point) Let $0 < \gamma < 1$ be a mass parameter and let $d\mu = (1-\gamma)\frac{d\theta}{2\pi} + \gamma\delta(\theta - 0)$, where $\delta(\theta-0)$ is the delta function at $\theta = 0$. To compute the moments,

$$\begin{aligned} c_n &= \int e^{-in\theta}\left((1-\gamma)\frac{d\theta}{2\pi} + \gamma\delta(\theta-0)\right) \\ &= (1-\gamma)\delta_{n,0} + \gamma \\ &= \begin{cases} 1 & \text{if } n = 0 \\ \gamma & \text{if } n \geq 1. \end{cases} \end{aligned}$$

Define P_j as the $j \times j$ matrix with all elements equal to j^{-1}. That is,

$$P_j = \begin{pmatrix} j^{-1} & \dots & \dots & j^{-1} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ j^{-1} & \dots & \dots & j^{-1} \end{pmatrix}.$$

Let L_n be the transpose of the $(n+1)$ Toeplitz matrix (i.e., $L_n = (T^{(n+1)})^T$).

It can be shown that

$$L_n = (1-\gamma)I + (n+1)\gamma P_{n+1},$$

where I is the $n \times n$ identity matrix. Therefore, L_n has the eigenvalue $1-\gamma$ with multiplicity n and the eigenvalue $1+n\gamma$ with multiplicity 1. Since L_n is the transpose of the Toeplitz matrix, whose determinant is real, the determinant of L_n is equal to D_n:

$$\begin{aligned} D_n &= \det(T^{(n+1)}) \\ &= \det(L_n) \\ &= (1-\gamma)^n(1+n\gamma). \end{aligned}$$

Using (3.37), we have

$$\begin{aligned} 1-|\alpha_n|^2 &= \frac{D_{n+1}D_{n-1}}{D_n^2} \\ &= \frac{(1-\gamma)^n(1+n\gamma+\gamma)(1+n\gamma-\gamma)}{(1+n\gamma)^2}, \end{aligned}$$

which simplifies to

$$|\alpha_n| = \frac{\gamma}{1+n\gamma}.$$

It can also be shown that $L_n^{-1} = (1-\gamma)^{-1}(I-P_{n+1}) + (1+n\gamma)^{-1}P_{n+1}$. From Theorem 3.19, we have

$$\Phi_n = z^n + \sum_{k=0}^{n-1} a_k^{(n)} z^k,$$

where the vector $a^{(n)}$ can be computed from

$$a^{(n)} = \langle \delta_n, L_n^{-1}\delta_n \rangle^{-1} L_n^{-1}\delta_n.$$

After some computations, we find that

$$a^{(n)} = \begin{pmatrix} \frac{-\gamma}{1+(n-1)\gamma} \\ \vdots \\ \frac{-\gamma}{1+(n-1)\gamma} \\ 1 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \Phi_n(z) &= z^n - \frac{\gamma}{1+(n-1)\gamma} \sum_{j=0}^{n-1} z^j \\ &= z^n - \alpha_{n-1}(z^{n-1} + z^{n-2} + \ldots + z + 1). \end{aligned}$$

Because $\alpha_n = -\overline{\Phi_{n+1}(0)}$,

$$\alpha_n = \frac{\gamma}{1+n\gamma}.$$

It follows that the reverse polynomials are

$$\Phi_n^* = 1 - \alpha_{n-1}(z + z^2 + \ldots + z^n),$$

as α_n is real. To get the orthonormal polynomials, we need to compute length of Φ_n:

$$\begin{aligned} ||\Phi_n||^2 &= \langle \Phi_n, \Phi_n \rangle \\ &= \langle \Phi_n, z^n - \alpha_{n-1}(z^{n-1} + z^{n-2} + \ldots + z + 1) \rangle \\ &= \langle \Phi_n, z^n \rangle \\ &= \langle z^n - \alpha_{n-1}(z^{n-1} + z^{n-2} + \ldots + z + 1), z^n \rangle. \end{aligned}$$

Expanding the inner product, notice that the first term is of length 1, and the lower order terms pick up only the Dirac delta n times, as the Lebesgue measure integrates to zero. Hence

$$||\Phi_n||^2 = 1 - \alpha_{n-1} n\gamma.$$

We can plug in α_{n-1} to obtain

$$\begin{aligned} ||\Phi_n||^2 &= 1 - n\gamma \left(\frac{\gamma}{1+(n-1)\gamma} \right) \\ &= 1 - \gamma \left(\frac{n}{\frac{1}{\gamma} - 1 + n} \right) \\ &= 1 - \gamma \left(1 + \frac{-(\frac{1}{\gamma} - 1)}{\frac{1}{\gamma} - 1 + n} \right) \\ &= 1 - \gamma + \mathcal{O}\left(\frac{1}{n} \right). \end{aligned}$$

Noticing that $\alpha_n \to 0$, we find

$$\begin{aligned} \varphi_n^* &= \frac{1 - \alpha_{n-1}(z + z^2 + \ldots + z^n)}{\sqrt{1 - \gamma + \mathcal{O}(\frac{1}{n})}} \\ \lim_{n\to\infty} \varphi_n^* &= \frac{1}{(1-\gamma)^{\frac{1}{2}}} \\ &= D(z)^{-1}. \end{aligned}$$

This result is consistent with the density of the a.c. part equal to the modulus squared of the Szegö function (i.e., $|D(e^{i\theta})|^2 = w(\theta)$). Finally, because the $n \geq 1$ moments are constants γ, the Carathédory function is a simple geometric sum:

$$\begin{aligned} F(z) &= 2 + 2\gamma \sum_{j=1} z^j \\ &= 1 + 2\gamma \frac{z}{1-z}. \end{aligned}$$

It follows that the Schur function f is

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{F-1}{F+1} \\ &= \frac{\gamma}{1-(1-\gamma)z}. \end{aligned}$$

Example 4.4. (Single Nontrivial Moment) Let $w(\theta) = (1 - \cos\theta)$. The case when $d\mu(\theta) = (1 - a\cos\theta)\frac{d\theta}{2\pi}$ where $0 < a < 1$ is investigated in Simon (2005*b*).

The tools used to analyze this case is essentially the same as the case for $a = 1$. Notice there is no singular part to the measure, because the absolutely continuous part integrates to 1. We compute the moments:

$$\begin{aligned}
c_n &= \int_0^{2\pi} e^{-in\theta}(1-\cos\theta)\frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} e^{-in\theta}\frac{d\theta}{2\pi} - \int_0^{2\pi} e^{-in\theta}\cos\theta\frac{d\theta}{2\pi} \\
&= \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n\geq 1 \end{cases} - \begin{cases} 0 & \text{if } n=0 \\ -\frac{1}{2} & \text{if } n=1 \\ 0 & \text{if } n\geq 2 \end{cases} \\
&= \begin{cases} 1 & \text{if } n=0 \\ -\frac{1}{2} & \text{if } n=1 \\ 0 & \text{if } n\geq 2. \end{cases}
\end{aligned}$$

Therefore, the Toeplitz matrix T_n is tridiagonal. Let J_n be the $n \times n$ matrix equal to $2T^{(n)}$. We have

$$J_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & & -1 & 2 \end{pmatrix}.$$

Let $d_n = \det(J_n)$, expanding along the first row and then expanding the minor of (-1) along the first column, we obtain the following recursive relation:

$$d_n = 2d_{n-1} - d_{n-2}.$$

With the initial conditions $d_1 = 2$ and $d_2 = 3$, we can show by induction that

$d_n = n + 1$:

$$\begin{aligned} d_n &= 2d_{n-1} - d_{n-2} \\ &= 2(n-1+1) - (n-2+1) \\ &= n+1. \end{aligned}$$

Thus, the Toeplitz determinant is

$$\begin{aligned} D_n &= \det(\frac{1}{2}J_{n+1}) \\ &= d_{n+1}\left(\frac{1}{2}\right)^{n+1} \\ &= \left(\frac{1}{2}\right)^{n+1}(n+2). \end{aligned}$$

We use standard Wronskian methods to find the inverse of J_n, which we'll need to compute L_n^{-1} in Theorem 3.19 and hence the coefficients of Φ_n. We solve:

$$d_n = 2d_{n-1} - d_{n-2}.$$

Using Wronskian methods with the appropriate boundary conditions, we obtain

$$\begin{aligned} (J_n^{-1})_{ij} &= (n+1)^{-1}(\min(i,j)+1)(n-\max(i,j)) \\ &= \frac{1}{n+1}\begin{pmatrix} n & (n-1) & \dots & 1 \\ (n-1) & & & 2 \\ \vdots & & & \vdots \\ 1 & 2 & \dots & n \end{pmatrix}. \end{aligned}$$

Note that the index (i,j) ranges from 0 to $n-1$. Also, observe that this matrix is equal to its transpose, so

$$\begin{aligned} J_n &= 2T^{(n)} \\ &= 2L_{n-1}^T \\ J_n^{-1} &= \frac{1}{2}(L_{n-1}^{-1})^T \\ L_{n-1}^{-1} &= 2(J_n^{-1})^T. \end{aligned}$$

Using the matrix elements above, we find that

$$\begin{aligned}
\langle \delta_0, L_n^{-1}\delta_n \rangle &= \frac{2}{n+2} \\
\langle \delta_n, L_n^{-1}\delta_n \rangle &= \frac{2(n+1)}{n+2}.
\end{aligned}$$

Therefore, from Theorem 3.19, we have the Verblunsky coefficients

$$\begin{aligned}
\alpha_{n-1} &= -\overline{\frac{1}{n+1}} \\
\alpha_n &= \frac{-1}{n+2}.
\end{aligned}$$

To find the monic OP's, let $P(z)$ be a degree n polynomial on the unit circle. That is

$$P(z) = \sum_{j=0}^{n} \eta_j z^j.$$

For $l = 0, 1, \ldots, n$,

$$\begin{aligned}
\langle z^l, P(z) \rangle &= \int \overline{z^l}\Big(\sum_{j=0}^{n} \eta_j z^j\Big) d\mu \\
&= \begin{cases} \eta_0 - \frac{1}{2}\eta_1 & \text{if } l = 0 \\ \int \big(\sum_{j=0}^{n} \eta_j z^{j-l}\big) d\mu & \text{if } 1 \le l \le n \end{cases} \\
&= \begin{cases} \eta_0 - \frac{1}{2}\eta_1 & \text{if } l = 0 \\ \eta_l - \frac{1}{2}(\eta_{l+1} + \eta_{l-1}) & \text{if } 1 \le l \le n. \end{cases}
\end{aligned}$$

Now, let $P(z) = \Phi_n$. Because Φ_n is orthogonal to $1, z, \ldots, z^{n-1}$ we have

$$\begin{aligned}
2\eta_0 &= \eta_1 \\
\eta_{l+1} &= 2\eta_l - \eta_{l-1}.
\end{aligned}$$

Inductively, we find that $\eta_j = \eta_0(j+1)$. Because Φ_n is monic, $\eta_n = 1$, so

$\eta_0 = \frac{1}{n+1}$, which gives us

$$\Phi_n = \frac{1}{n+1}\sum_{j=0}^{n}(j+1)z^j.$$

Notice that this formula also verifies that $\alpha_n = -\frac{1}{n+2}$. To find the orthonormal polynomials,

$$||\Phi_n||^2 = \frac{D_n}{D_{n-1}} = \frac{(\frac{1}{2})^{n+1}(n+2)}{(\frac{1}{2})^n(n+1)} = \frac{1}{2}\left(\frac{n+1}{n+2}\right),$$

and thus the orthonormal polynomials are:

$$\begin{aligned}\varphi_n &= \sqrt{\frac{2}{(n+1)(n+2)}}\sum_{j=0}^{n}(j+1)z^j \\ \varphi_n^* &= \sqrt{\frac{2}{(n+1)(n+2)}}\sum_{j=0}^{n}(n-j+1)z^j.\end{aligned}$$

To compute the Szegö function, note

$$\begin{aligned}\lim_{n\to\infty}\varphi_n^* &= \lim_{n\to\infty}\sqrt{\frac{2}{(n+1)(n+2)}}\sum_{j=0}^{n}(n-j+1)z^j \\ &= \lim_{n\to\infty}\sqrt{\frac{2}{(n+1)(n+2)}}\sum_{j=0}^{n}(n+1)z^j - \lim_{n\to\infty}\sqrt{\frac{2}{(n+1)(n+2)}}\sum_{j=0}^{n}jz^j.\end{aligned}$$

As we take $n \to \infty$, the first term is a geometric sum. From differentiating identities for

$$\frac{z}{(1-z)^2}$$

we get that the second term is equal to

$$\lim_{n\to\infty}\sqrt{\frac{2}{(n+1)(n+2)}}\frac{z+(-1+n(z-1))z^{n+1}}{(z-1)^2},$$

which tends to zero as $n \to \infty$ because $|z| < 1$ for the Szegö function. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \varphi_n^* &= \sqrt{2}\sum_{j=0}^{\infty} z^j \\
&= \sqrt{2}\frac{1}{1-z} \\
D(z) &= \frac{1-z}{\sqrt{2}}.
\end{aligned}$$

Finally, since the moments for $n \geq 2$ are zero, the Carathédory function is a finite sum:

$$\begin{aligned}
F(z) &= 1 + 2c_1 z = 1 - z \\
f(z) &= \frac{1}{z}\frac{F-1}{F+1} = \frac{1}{z-2}.
\end{aligned}$$

In fact, the density $w(\theta)$ from this case is connected to a classic example in quantum mechanics (Griffiths 2005). The m-th level wave-function for a particle in the box of length a $(0 < x < a)$ is

$$\psi_m = \sqrt{\frac{2}{a}}\sin\left(\frac{m\pi x}{a}\right).$$

If we let the box have length $a = 2\pi$, then

$$\psi_m = \sqrt{\frac{1}{\pi}}\sin\left(\frac{m\theta}{2}\right).$$

The properly normalized probability density $|\psi|^2$ is

$$w(\theta) = 2\sin^2\left(\frac{m\theta}{2}\right).$$

Using the double angle identity $\sin^2(\theta) = 1 - \frac{\cos(2\theta)}{2}$, we see that

$$w(\theta) = 1 - \cos(m\theta),$$

which is precisely the single nontrivial moment case for OPUC, where the nontrivial moment occurs at the m-th moment.

4.2 New Classes of OPUC

Before we present the examples of OPUC, the following theorem relates even Verblunsky coefficients with odd moments.

Theorem 4.5. *We have $c_j = 0$ for odd j if and only if $\alpha_j = 0$ for even j.*

Proof. Equation (3.38) states that

$$\alpha_n = (-1)^n \frac{D_n(e^{-i\theta} d\mu)}{D_n(d\mu)}. \tag{4.1}$$

For the forward direction, we have that $\alpha_{2j} = 0$ for all integer $j \geq 0$ and we use strong induction. Since $c_1 = \alpha_0$, the base case is trivially satisfied as they are both equal to 0. Assume that the result holds for all odd moments $1, 3, \ldots, 2n-1$, we prove that it is true for $2n+1$. Note that (4.1) becomes

$$\begin{aligned} \alpha_{2n} &= (-1)^{2n} \frac{D_{2n}(e^{-i\theta} d\mu)}{D_{2n}(d\mu)} \\ 0 &= \frac{D_{2n}(e^{-i\theta} d\mu)}{D_{2n}(d\mu)}. \end{aligned}$$

The denominator is a Toeplitz determinant, which is non-zero because the moments come from a non-trivial probability measure on $\partial\mathbb{D}$. Therefore, we know that $\alpha_{2n} = 0$ if and only if $D_{2n}(e^{-i\theta} d\mu) = 0$ for all n. With the induction hypothesis, the numerator is the determinant

$$D_{2n}(e^{-i\theta} d\mu) = \det \begin{pmatrix} 0 & c_2 & 0 & c_4 & \ldots & 0 & c_{2n} & c_{2n+1} \\ c_0 & 0 & c_2 & 0 & & c_{2n-2} & 0 & c_{2n} \\ & & & \vdots & & & & \vdots \\ & & \vdots & & & \vdots & & \\ & \vdots & & & & \vdots & & \\ & & & & & & & \\ \overline{c_{2n-2}} & 0 & & \vdots & & & 0 & c_2 \\ 0 & \overline{c_{2n-2}} & 0 & & & 0 & c_0 & 0 \end{pmatrix}.$$

We need not assume anything about the values of the even moments. We'll keep the indices just for counting purposes. Next, expand the determinant along the top row, starting from the c_{2n+1} place, we have

$$\begin{aligned} D_{2n}(e^{-i\theta}d\mu) &= (-1)^{2n}c_{2n+1}D_{2n-1}(d\mu) + \sum_{\substack{i=2 \\ i \text{ even}}}^{2n} \pm(-1)\text{Det of Minors} \\ &= c_{2n+1}D_{2n-1}(d\mu) + \sum_{\substack{i=2 \\ i \text{ even}}}^{2n} \pm(-1)\text{Det of Minors}. \end{aligned}$$

As mentioned before, $D_{2n-1}(d\mu)$ is non-zero. Therefore, if we show that all the determinant of minors are equal to zero, then it follows that $D_{2n}(e^{-i\theta}d\mu) = 0$ if and only if $c_{2n+1} = 0$. To see that the determinant of those minors are all zero, we simply use a counting argument in linear algebra. For counting purposes, re-label the even moments $c_2, c_4, \ldots, c_{2n}$ as $k_1, k_2, \ldots, k_n$. Note that the determinant is

$$D_{2n}(e^{-i\theta}d\mu) = \begin{vmatrix} 0 & k_1 & 0 & k_2 & \ldots & 0 & k_n & c_{2j+1} \\ k_0 & 0 & k_1 & 0 & & k_{n-1} & 0 & k_n \\ & & & \vdots & & & & \vdots \\ & & \vdots & & & \vdots & & \\ & \vdots & & & & \vdots & & \\ & & & & & & & \\ k_{n-1} & 0 & & \vdots & & & 0 & k_1 \\ 0 & k_{n-1} & 0 & & & 0 & k_0 & 0 \end{vmatrix}.$$

This is a $2n+1 \times 2n+1$ matrix, so from the second row to the last row, there are $2n$ row vectors of length $2n+1$. There are n row vectors that have $(n+1)$ zeros and n non-zero k_i's. When we calculate the determinant of the minors of $k_1, k_2, \ldots, k_n$, we remove one non-zero element from those vectors. For each minor, we are left with n row vectors of length $2n$, all of which have only $(n-1)$ non-zero elements. By linear dependence, all the other minors (except the c_{2j+1} minor) have determinant zero. This completes the forward direction. A similar proof gives the other direction. □

Example 4.6. (Shifted, single non-zero Verblunsky coefficient) Let the Verblunsky coefficients be the list $\{0, \frac{1}{2}, 0, 0, 0, \ldots\}$.[1] The monic orthogonal polynomials can be directly computed via the Szegö recursion, with

$$\begin{aligned}\Phi_0 &= 1\\ \Phi_1 &= z\\ \Phi_2 &= z^2 - \frac{1}{2}\\ \ldots\end{aligned}$$

so that

$$\Phi_n = \begin{cases} 1 & \text{if } n = 0\\ z & \text{if } n = 1\\ z^2 - \frac{1}{2}z^{n-2} & \text{if } n \geq 2.\end{cases}$$

Furthermore, we have that: $\Phi_n^* = 1 - \frac{1}{2}z^2$ for $n \geq 2$. Since we know the α_j's, we can also compute the length of the OP's for $n \geq 2$:

$$||\Phi_n||^2 = \prod_{j=0}^{n-1} 1 - |\alpha_j|^2 = \frac{3}{4}.$$

We can also directly compute the Toeplitz determinants:

$$\begin{aligned}D_n &= \prod_{j=0}^{n-1}(1 - |\alpha_j|^2)^{n-j}\\ &= \begin{cases} 1 & \text{if } n = 1\\ \frac{3}{4} & \text{if } n \geq 2.\end{cases}\end{aligned}$$

Thus, we have that

$$\lim_{n\to\infty} \varphi_n^* = \varphi_n^* = \frac{2 - z^2}{\sqrt{3}}.$$

[1] We use $\frac{1}{2}$ here for concreteness. It can easily be generalized to $|\alpha| < 1$, as all the methods here still apply.

We may then compute the Szegö function and obtain the density:

$$
\begin{aligned}
D(z)^{-1} &= \lim_{n\to\infty} \varphi_n^* = \frac{2-z^2}{\sqrt{3}} \\
D(z) &= \frac{\sqrt{3}}{2-z^2} \\
w(\theta) &= |D(e^{i\theta})|^2 \\
&= \left(\frac{3}{2-e^{2i\theta}}\right)\left(\frac{3}{2-e^{-2i\theta}}\right) \\
&= \frac{3}{5-4\cos(2\theta)}.
\end{aligned}
$$

There is no singular part to the measure because the a.c. part integrates to 1. To see this, we use a standard Weistrauss substitution method. We first derive a more general definite integral. Let

$$
\begin{aligned}
I &= \int_0^\pi \frac{d\theta}{\alpha+\beta\cos(\theta)} \\
t &= \tan\left(\frac{\theta}{2}\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\theta}{2} &= \arctan(t) \\
\cos(\theta) &= \frac{1-t^2}{1+t^2} \\
d\theta &= \frac{2}{1+t^2}dt.
\end{aligned}
$$

We can now evaluate the integral I:

$$\begin{aligned}
I &= \int_0^\infty \frac{\frac{2}{1+t^2}dt}{\alpha+\beta\frac{1-t^2}{1+t^2}} \\
&= 2\int_0^\infty \frac{dt}{\alpha(1+t^2)+\beta(1-t^2)} \\
&= 2\int_0^\infty \frac{dt}{(\alpha+\beta)+(\alpha-\beta)t^2} \\
&= \frac{2}{\alpha^2-\beta^2}\arctan(\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}t)|_0^\infty \\
&= \frac{\pi}{\alpha^2-\beta^2}.
\end{aligned}$$

Now, note that

$$\begin{aligned}
\int_0^{2\pi} w(\theta)\frac{d\theta}{2\pi} &= \int_0^{2\pi}\frac{3}{5-4\cos(2\theta)}\frac{d\theta}{2\pi} \\
&= \int_0^{\pi}\frac{3}{5-4\cos(\theta)}\frac{d\theta}{2\pi},
\end{aligned}$$

which by the formula above gives

$$\int_0^{2\pi} w(\theta)\frac{d\theta}{2\pi} = 1.$$

Now, we can compute the Schur function f from its continued fraction expansion, noting that from Geronimus' Theorem the Schur parameters are equal to the Verblunsky coefficients (i.e., $\gamma_j = \alpha_j$):

$$\begin{aligned}
f &= \gamma_0 + \frac{1-|\gamma_0|^2}{\overline{\gamma_0}+\frac{1}{z\gamma_1+\frac{z(1-|\gamma_1|^2)}{z\gamma_2+\ldots}}} \\
&= \frac{1}{\frac{1}{\frac{1}{2}z+\frac{z(1-\frac{1}{4})}{\frac{1}{2}+z^{-\infty}}}} \\
&= \frac{z}{2},
\end{aligned}$$

since f is defined strictly within the unit disk, so $z^{-\infty}$ is complex infinity. This also gives F,

$$\begin{aligned} F &= \frac{1+zf}{1-zf} \\ &= \frac{1+\frac{z^2}{2}}{1-\frac{z^2}{2}} \\ &= \frac{2+z^2}{2-z^2}. \end{aligned}$$

Taking the real part of F and taking the radial limit at $e^{i\theta}$, we indeed recover the correct density $w(\theta)$:

$$\begin{aligned} \mathrm{Re}F(e^{i\theta}) &= \frac{1}{2}\left(\frac{2+e^{2i\theta}}{2-e^{2i\theta}} + \frac{2+e^{-2i\theta}}{2-e^{-2i\theta}}\right) \\ &= \frac{3}{5-4\cos(2\theta)} \\ &= w(\theta). \end{aligned}$$

Next we compute even the moments. For even n,

$$c_n = \int_0^{2\pi} e^{-in\theta} \frac{3}{5-4\cos(2\theta)} \frac{d\theta}{2\pi}.$$

We know that the odd moments are zero by Theorem 4.5. Define $f(z)$ by

$$\begin{aligned} f(z) &= \frac{3}{5-2(z^2+z^{-2})} \\ &= \frac{1}{2z^2-1} - \frac{2}{z^2-2}. \end{aligned}$$

Then $f(e^{i\theta}) = \frac{3}{5-4\cos(2\theta)} = w(\theta)$. For $|z| < \sqrt{2}$,

$$\begin{aligned} \frac{-2}{z^2-2} &= \frac{1}{1-\frac{z^2}{2}} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^k} z^{2k}. \end{aligned}$$

For $|z| > \frac{1}{\sqrt{2}}$,

$$\begin{aligned} \frac{1}{2z^2-1} &= \frac{1}{2z^2}\frac{1}{1-\frac{1}{2z^2}} \\ &= \sum_{k=0}^{\infty}(\frac{1}{2z^2})^k\frac{1}{2z^2} \\ &= \sum_{k=1}^{\infty}\frac{1}{2^k}z^{-2k}. \end{aligned}$$

Since $\frac{1}{\sqrt{2}} < |z| = 1 < \sqrt{2}$,

$$\begin{aligned} \frac{3}{5-4\cos(2\theta)} &= 1+\sum_{k=1}\frac{1}{2^k}(z^{2k}+z^{-2k}) \\ &= 1+\sum_{k=1}\frac{1}{2^{k-1}}\cos(2k\theta). \end{aligned}$$

We can now compute the moments using the orthogonality of the cosines

$$\begin{aligned} c_n &= \int_0^{2\pi} e^{-in\theta}(1+\sum_{k=1}\frac{1}{2^{k-1}}\cos(2k\theta))\frac{d\theta}{2\pi} \\ &= \int_0^{2\pi}\cos(n\theta)\frac{2}{2^{\frac{n}{2}}}\cos(n\theta)\frac{d\theta}{2\pi} \\ &= \left(\frac{1}{2}\right)^{\frac{n}{2}}. \end{aligned}$$

Example 4.7. (Linear Density) Let $w(\theta) = \frac{\theta}{\pi}$. Notice that since $\int w(\theta)\frac{d\theta}{2\pi} = 1$, we have $d\mu_s = 0$. The moments can be directly computed using integration by parts:

$$\begin{aligned} c_n &= \int e^{-in\theta}\frac{\theta}{\pi}\frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi^2}\left[i\frac{\theta}{n}e^{-in\theta}+\frac{1}{n}e^{-in\theta}\right]_{\theta=0}^{\theta=2\pi} \\ &= \frac{i}{n\pi}. \end{aligned}$$

Hence, the Carathédory function F is

$$\begin{aligned} F &= 1 + \sum_{n=1} 2c_n z^n \\ &= 1 + \frac{2i}{\pi} \sum_{n=1} \frac{1}{n} z^n. \end{aligned}$$

At $\partial\mathbb{D}$,

$$\begin{aligned} F(e^{i\theta}) &= 1 + \frac{2i}{n\pi} \sum_{n=1} \cos(n\theta) + i\sin(n\theta) \\ \mathrm{Re}F(e^{i\theta}) &= 1 - \frac{2}{\pi} \sum_{1} \frac{1}{n} \sin(n\theta), \end{aligned}$$

which we recognize is the Fourier series for $w(\theta) = \frac{\theta}{\pi}$, as the (odd) Fourier series a_n are

$$\begin{aligned} a_n &= \frac{1}{\pi} = \int \frac{\theta}{\pi} \sin(n\theta) d\theta = \frac{-2}{n\pi} \\ a_0 &= \frac{1}{2\pi} \int \frac{\theta}{\pi} d\theta = 1. \end{aligned}$$

Alternatively, we can recovering the a.c. density from the Carathédory function using complex logarithms as follows. Let $s = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots$. Then

$$\begin{aligned} \frac{d}{dz}s &= 1 + z + z^2 + \cdots \\ &= \frac{1}{1-z} \end{aligned}$$

with radius of convergence $|z| < 1$. Hence

$$\begin{aligned} s &= \int \frac{d}{dz}s \\ &= -\log(1-z), \end{aligned}$$

so we have that the Carathédory function, defined within the unit disk, is equal to

$$F(z) = 1 - \frac{2i}{\pi} \log(1-z).$$

Approaching the boundary $\partial\mathbb{D}$, we have

$$\begin{aligned} F(e^{i\theta}) &= 1 - \frac{2i}{\pi}\log(1-e^{i\theta}) \\ &= 1 - \frac{2i}{\pi}\log(e^{\frac{i\theta}{2}}(e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}})) \\ &= 1 - \frac{2i}{\pi}\log\left(e^{\frac{i\theta}{2}}(-2i\sin(\frac{\theta}{2})\right). \end{aligned}$$

Noting that $\log(2i) = \frac{-i\pi}{2} + \log 2$,

$$\begin{aligned} &= 1 - \frac{2i}{\pi}\left(\frac{i\theta}{2} + (\frac{-i\pi}{2} + \log 2) + \log(\sin(\frac{\theta}{2}))\right) \\ &= 1 + \frac{\theta}{\pi} - 1 - \frac{2i}{\pi}\log(2) - \frac{2i}{\pi}\log(\sin(\frac{\theta}{2})) \\ &= \frac{\theta}{\pi} - \frac{2i}{\pi}\left[\log(2) + \log\left(\sin\left(\frac{\theta}{2}\right)\right)\right]. \end{aligned}$$

For $0 \le \theta \le 2\pi$, $0 \le \frac{\theta}{2} \le \pi$, the sine function is positive, so the logarithm is real (approaching from the right), which makes the last term purely imaginary. Hence, taking the real part, we indeed recover the density $w(\theta)$:

$$\begin{aligned} \mathrm{Re}F(e^{i\theta}) &= \frac{\theta}{\pi} \\ &= w(\theta). \end{aligned}$$

Example 4.8. (Single non-zero Verblunsky Coefficient, General Case) Let $\alpha_n = \delta_{N,n}\gamma$, where $\gamma \in \mathbb{R}$, $|\gamma| < 1$, $N \in \mathbb{Z}^+$. Requiring γ to be real simplifies the algebra when integrating for the moments, but the results in this example hold for any $\gamma \in \mathbb{D}$. Also note that the case $N = 0$ corresponds to the Bernstein-Szegö polynomials derived in Simon (2005*b*), so we are deriving a more general expression. For non-negative integer N, we have

$$\alpha_n = \begin{cases} 0 & \text{if } n \neq N \\ \gamma & \text{if } n = N. \end{cases}$$

From the Szegö recursion, we can directly compute the monic OP's, getting

$$\Phi_n(z) = \begin{cases} z^n & \text{if } n < N \\ z^n - \gamma z^{n-(N+1)} & \text{if } n \geq N \end{cases}$$

$$\Phi_n^*(z) = \begin{cases} 1 & \text{if } n < N \\ 1 - \gamma z^{(N+1)} & \text{if } n \geq N. \end{cases}$$

The length of the OP's also directly follow from the Verblunsky coefficients:

$$\begin{aligned} ||\Phi_n||^2 &= \prod_0^{n-1} 1 - |\alpha_j|^2 \\ &= \begin{cases} 1 & \text{if } n \leq N \\ 1 - \gamma^2 & \text{if } n > N. \end{cases} \end{aligned}$$

Therefore, we have that for $n > N$,

$$\begin{aligned} \varphi_n^* &= \frac{1 - \gamma z^{N+1}}{\sqrt{1-\gamma^2}} \\ &= \lim_{n\to\infty} \varphi_n^* \\ &= D(z)^{-1}. \end{aligned}$$

Thus,

$$D(z) = \frac{\sqrt{1-\gamma^2}}{1 - \gamma z^{N+1}}.$$

From the Szegö function, we are able to recover the a.c. density:

$$\begin{aligned} w(\theta) &= |D(e^{i\theta})|^2 \\ &= \frac{1-\gamma^2}{(1-\gamma e^{i(N+1)\theta})(1-\gamma e^{-i(N+1)\theta})} \\ &= \frac{1-\gamma^2}{1-\gamma e^{-i(N+1)\theta} - \gamma e^{i(N+1)\theta} + \gamma^2} \\ &= \frac{1-\gamma^2}{1+\gamma^2 - 2\gamma\cos(N+1)\theta}. \end{aligned}$$

At this point, it is not clear that there is no singular part to the measure. However, as we will see soon, the a.c. part integrates to 1, so that $d\mu_s = 0$. With a bit of complex analysis, we now write $w(\theta)$ as a Taylor series. Let

$$f(z) = \frac{1-\gamma^2}{1+\gamma^2-\gamma(z^{N+1}+z^{-(N+1)})}.$$

Note that $f(e^{i\theta}) = w(\theta)$. We may rewrite $f(z)$ as

$$\begin{aligned} f(z) &= \frac{\gamma}{z^{(N+1)}-\gamma} + \frac{1}{1-\gamma z^{(N+1)}} \\ &= \frac{\gamma}{z^{N+1}(1-\frac{\gamma}{z^{N+1}})} + \frac{1}{1-\gamma z^{N+1}}. \end{aligned}$$

Let $z = e^{i\theta}$. We recognize the two terms above as geometric series, with radius of convergence $|\frac{\gamma}{z^{N+1}}| < 1$ and $|\gamma| < |z|^{N+1}$. Since $|z| = 1$ and $|\gamma| < 1$, both these cases are satisfied:

$$\begin{aligned} f(z) &= \frac{\gamma}{z^{N+1}} \sum_{k=0}^{\infty} \gamma^k (z^{N+1})^{-k} + \sum_{k=0}^{\infty} \gamma^k (z^{N+1})^k \\ &= 1 + \sum_{k=1}^{\infty} \gamma^k (z^{(N+1)k} + z^{-(N+1)k}) \\ f(e^{i\theta}) &= 1 + 2\sum_{k=1}^{\infty} \gamma^k \cos(N+1)k\theta \\ &= w(\theta). \end{aligned}$$

Since all the cosine terms integrate to 0 with respect to the Lebesgue measure, we get that the integral of $w(\theta)$ is 1, so there is no singular part. From integrating $w(\theta)$, we also get that the moments are decreasing as powers of γ (i.e., $\gamma, \gamma^2, \gamma^3, \ldots$), which is another argument for why there is no singular part. Moreover, the interesting piece is that there are exactly N zeros in between every non-zero moment! For example, if $N = 2$, then the moments would be $\{\gamma, 0, 0, \gamma^2, 0, 0, \gamma^3, \ldots\}$. The Schur function is

$$f(z) = \gamma z^N,$$

which gives

$$\begin{aligned} F(z) &= \frac{1+zf}{1-zf} \\ &= \frac{1+\gamma z^{N+1}}{1-\gamma z^{N+1}}; \end{aligned}$$

the reader can check that, indeed, $\mathrm{Re}F(e^{i\theta}) = w(\theta)$. We can also compute the Toeplitz determinant as:

$$\begin{aligned} D_n &= \prod_{j=0}^{n-1}(1-|\alpha_j|^2)^{n-j} \\ &= \begin{cases} 1 & \text{if } n \le N \\ (1-\gamma^2)^{n-N} & \text{if } n > N. \end{cases} \end{aligned}$$

Example 4.9. (Rotated, Single-Inserted Mass Point) We begin by defining the Verblunsky coefficients as $\alpha_j = \frac{(-1)^{j+1}}{j+2}$. The list of Verblunsky coefficients is simply $\{\frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \ldots\}$. First, we claim that

$$\begin{aligned} \Phi_n(z) &= z^n - \frac{1}{n+1}\sum_{j=0}^{n-1}(-1)^{n-j}z^j \\ \Phi_n^*(z) &= 1 - \frac{1}{n+1}\sum_{j=1}^{n}(-1)^j z^j. \end{aligned}$$

We prove this using induction and the Szegö recursion, since we have the explicit formula for α_j. The base case is trivially satisfied. Assuming the induction hypothesis for n, we complete the induction by computing $\Phi_{n+1}(z)$ via the Szegö recursion:

$$\begin{aligned} z\Phi_n(z) \;-\; \overline{\alpha_n}\Phi_n^*(z) &= z\left(z^n - \frac{1}{n+1}\sum_{j=0}^{n-1}(-1)^{n-j}z^j\right) - \frac{(-1)^{j+1}}{j+2}\left(1 - \frac{1}{n+1}\sum_{j=1}^{n}(-1)^j z^j\right) \\ &= z^{n+1} - \frac{1}{n+1}\sum_{j=1}^{n}(-1)^{n-j+1}z^j - \frac{(-1)^{n+1}}{n+2} + \frac{1}{(n+1)(n+2)}\sum_{j=1}^{n}(-1)^{n+j+1}z^j \\ &= z^{n+1} - \frac{(-1)^{n+1}}{n+2} + \sum_{j=1}^{n} z^j\left(\frac{(-1)^{n+j+1}}{(n+1)(n+2)} - \frac{(-1)^{n-j+1}}{n+1}\right). \end{aligned}$$

Note that the constant terms are already equal to $-\overline{\alpha_{n-1}}$. Because $n+j$ and $n-j$ are congruent mod 2,

$$\begin{aligned}
&= z^{n+1} - \frac{(-1)^{n+1}}{n+2} + \sum_{j=1}^{n} z^j \left(\frac{(-1)^{n+1-j} - (-1)^{n+1-j}(n+2)}{(n+1)(n+2)} \right) \\
&= z^{n+1} - \frac{(-1)^{n+1}}{n+2} + \sum_{j=1}^{n} z^j (-\frac{(-1)^{n+1-j}}{n+2}) \\
&= z^{n+1} - \frac{1}{n+2} \sum_{j=0}^{n} z^j (-1)^{(n+1)-j} \\
&= \Phi_{n+1}(z).
\end{aligned}$$

Since we know the α_j's, we may also compute the length of the OP's:

$$\begin{aligned}
||\Phi_n||^2 &= \prod_{j=0}^{n-1} (1 - |\alpha_j|^2) \\
&= \prod_{j=0}^{n-1} \left(1 - \frac{1}{(j+2)^2} \right) \\
&= \prod_{j=0}^{n-1} \frac{(j+1)(j+3)}{(j+2)^2} \\
&= \frac{n+2}{2(n+1)}.
\end{aligned}$$

Therefore, the reverse orthonormal polynomials are

$$\begin{aligned}
\varphi_n^*(z) &= \sqrt{\frac{2(n+1)}{n+2}} (1 - \frac{1}{n+1} \sum_{j=1}^{n} (-1)^j z^j) \\
&= \sqrt{\frac{2(n+1)}{n+2}} - \sqrt{\frac{2}{(n+1)(n+2)}} \sum_{j=1}^{n} (-1)^j z^j.
\end{aligned}$$

Taking the limit as $n \to \infty$, we note that the sum in the second term is simply a geometric series whose sum is bounded, so the second term tends to zero. The first term simply tends to $\sqrt{2}$. Therefore,

$$\begin{aligned} D(z) &= \frac{1}{\sqrt{2}} \\ w(\theta) &= \frac{1}{2}. \end{aligned}$$

The moments are $c_n = 1/2$ for n even and $c_n = -1/2$ for n odd. To see this, we can use the rotation of measures formula in Simon (2005*b*) to obtain the rotated Verblunsky coefficients from a single inserted mass point of $\frac{1}{2}$ at $\theta = 0$. So the measure here is

$$d\mu = \frac{1}{2}\frac{d\theta}{2\pi} + \frac{\delta(\theta - \pi)}{2},$$

which isn't surprising because in the integration of the moments, the Lebesgue part integrates to 0, and the Delta function at π picks up 1/2 for even n and $-1/2$ for odd n. To go through this example, we could have alternatively started with this rotated measure and derived everything else from the rotated measure. We get that the Toeplitz determinant is

$$D_n(c) = \left(\frac{1}{2}\right)^n \left(1 + \frac{n}{2}\right).$$

Example 4.10. (Multiple Inserted Mass Points at the Roots of Unity)

Start by placing N point masses at the roots of unity, each with mass $\frac{\gamma}{N}$, so that the pure point adds up to $\gamma \in (0, 1)$. The case with two inserted mass points have been studied in Wong (2009). Let the rest of the measure, $1 - \gamma$, be the Lebesgue measure. That is:

$$d\mu = (1-\gamma)\frac{d\theta}{2\pi} + \frac{\gamma}{N}\left(\delta(\theta) + \delta(\theta - \frac{2\pi}{N}) + \ldots + \delta(\theta - (N-1)\frac{2\pi}{N})\right).$$

First, we compute the moments:

$$c_n = \int_0^{2\pi} e^{-in\theta} d\mu.$$

Note that the Lebesgue measure integrates to zero, and we only pick up the pure point parts of the measure. Let $\omega = e^{\frac{-2\pi i}{N}}$. We then have

$$c_n = \frac{\gamma}{N}(1^n + \omega^n + \omega^{2n} + \ldots + \omega^{(N-1)n}).$$

For $n = kN$ (n is a multiple of N), $k \in \mathbb{N}$, every term in the equation above is equal to 1 as $\omega^N = 1$, so we have

$$\begin{aligned} c_n &= \frac{\gamma}{N}(1 + 1 + \ldots + 1) \\ &= \gamma. \end{aligned}$$

For $n \neq kN$ (n is not a multiple of N), we have

$$\begin{aligned} c_n &= \frac{\gamma}{N}(1^n + \omega^n + \omega^{2n} + \ldots + \omega^{(N-1)n}) \\ &= \frac{\gamma}{N}\frac{\omega^{nN} - 1}{\omega^n - 1} \\ &= \frac{\gamma}{N}\frac{(\omega^N)^n - 1}{\omega^n - 1} \\ &= 0. \end{aligned}$$

Therefore, the moments are equal to zero if n is not equal to a multiple of N and equal to γ if n is a multiple of N. That is,

$$\{c_n\}_{n\geq 1} = \{\underbrace{0, 0, \ldots,}_{N-1} \gamma, 0, 0, \ldots, \gamma, \ldots\}.$$

For concreteness, let $N = 2$. The Verblunsky coefficients are then equal to

$$\{\alpha_j\}_{j\geq 0} = \{0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\}.$$

This can be obtained through (3.38) and the zeros are also obtainable through Theorem 4.5. Next, we want to compute the orthogonal polynomials. First, we do Gram-Schmidt for a few steps to observe the pattern.

$$
\begin{aligned}
\Phi_0 &= 1 \\
\Phi_1 &= z - \langle z, \Phi_0 \rangle \frac{\Phi_0}{||\Phi_0||^2} \\
&= z - \langle z, 1 \rangle \\
&= z - \int e^{-i\theta} d\mu \\
&= z - c_1 \\
&= z \\
\Phi_2 &= z^2 - \langle z^2, \Phi_1 \rangle \frac{\Phi_1}{||\Phi_1||^2} - \langle z^2, \Phi_0 \rangle \frac{\Phi_0}{||\Phi_0||^2} \\
&= z^2 - (\int z^{-1} d\mu) \frac{\Phi_1}{||\Phi_1||^2} - \int z^{-2} d\mu \\
&= z^2 - \gamma.
\end{aligned}
$$

In general, note that for this specific measure, $\int z^{-i+j} d\mu = c_{-i+j} \neq 0$ if and only if $-i + j$ is a multiple of $N = 2$. Therefore, it is not hard to see that for n that is not a multiple of N, $\Phi_n = z\Phi_{n-1}$. For example, we have

$$
\begin{aligned}
\Phi_3 &= z(z^2 - \gamma) \\
&= z^3 - \gamma z.
\end{aligned}
$$

Next, we prove by induction the general formula for the monic orthogonal polynomials. For tractability, let $\gamma = \frac{1}{2}$; the method will generalize to any $\gamma \in (0, 1)$. Specifically, we prove that for $k \in \mathbb{N}$,

$$
\begin{aligned}
\Phi_{2k} &= z^{2k} - \frac{1}{k+1}(z^{2k-2} + z^{2k-4} + \ldots + 1) \\
\Phi_{2k+1} &= \Phi_{2k} z.
\end{aligned}
$$

The base case is satisfied and the case for $2k+1$ is clear from the Gram-Schmidt argument above. Thus, it is enough to prove that this is true for $n = 2k$. First, note that the Verblunsky coefficient for odd j can be written as

$$\begin{aligned}\alpha_j &= \frac{j+1}{2}+1 \\ &= \frac{j+3}{2}.\end{aligned}$$

Using the Szegö recurrence, we have

$$\Phi_{2k} = z\Phi_{2k-1} - \overline{\alpha}_{n-1}\Phi^*_{2k-1}.$$

Applying the induction hypothesis, we have that

$$\begin{aligned}\Phi_{2k} &= z\Phi_{2k+1} - \frac{1}{\frac{(2k-1)+3}{2}}\Phi^*_{2k-1} \\ &= z(z)(z^{2k-2} - \frac{1}{k}(z^{2k-4} + \ldots + 1)) - \frac{1}{k+1}\Phi^*_{2k-1} \\ &= z^{2k} - \frac{1}{k}(z^{2k-2} + \ldots + z^2) - \frac{1}{k+1}(1 - \frac{1}{k}(z^2 + z^4 + \ldots + z^{2k-2})) \\ &= z^{2k} - \frac{1}{k}(z^{2k-2} + \ldots + z^2) - \frac{1}{k+1} - \frac{1}{k}\frac{1}{k+1}(z^{2k-2} + \ldots + z^2).\end{aligned}$$

Now, note that $\frac{1}{(k+1)k} - \frac{1}{k} = \frac{-1}{k+1}$, which gives us the desired result:

$$\Phi_{2k} = z^{2k} - \frac{1}{k+1}(z^{2k-2} + \ldots + z^2 + 1).$$

This immediately gives us the reverse polynomials Φ^*_n also. To obtain the length of the polynomials, and hence the orthonormal polynomials, we can simply use the formula

$$||\Phi_n||^2 = \prod_{j=0}^{n-1}(1 - |\alpha_j|^2)$$

which can be computed from the list of Verblunsky coefficients. Next, since the moments are equal to γ at every multiple of N and zero for all other n, the

Carathédory function F is a simple geometric series:

$$\begin{aligned} F(z) &= 1+\sum_{n=1} 2c_n z^n \\ &= 1+2\sum_{n=1} \gamma z^{nN} \\ &= 1+2\gamma\sum_{n=1} (z^N)^n \\ &= 1+2\gamma\frac{z^N}{1-z^N}. \end{aligned}$$

This also gives us the Schur function f:

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{F-1}{F+1} \\ &= \frac{1}{z}\frac{\gamma}{z^{-N}-(1-\gamma)} \end{aligned}$$

Example 4.11. ($0, \alpha, 0, \alpha, 0, \ldots$, continued fraction expansion) Begin with the Verblunsky coefficients as $0, \alpha, 0, \alpha, 0, \ldots$. That is,

$$\begin{aligned} \alpha_{2j+1} &= \alpha \\ \alpha_{2j} &= 0. \end{aligned}$$

This is a simple case of periodic Verblunsky coefficients. The key tool to analyze α_i's of this form is to look at the continued fraction expansion for the Schur function. Because α_i is periodic, the continued fraction expansion gives us

$$\begin{aligned} f(z) &= z\alpha+\frac{z(1-|\alpha|^2)}{\overline{\alpha}+\frac{1}{zf}} \\ &= z\alpha+\frac{z^2 f(1-|\alpha|^2)}{\overline{\alpha}zf+1} \\ &= \frac{z\alpha+z^2 f}{1+\overline{\alpha}zf}. \end{aligned}$$

Rearranging, we have:

$$\begin{aligned} f+\overline{\alpha}zf^2 &= z\alpha+z^2 f \\ \overline{\alpha}zf^2+(1-z^2)f-\alpha z &= 0. \end{aligned}$$

This is a quadratic function in f, which we can solve for in terms of its coefficients:

$$f = \frac{-(1-z^2)+\sqrt{(z^2-1)^2+4|\alpha|^2 z^2}}{2\overline{\alpha}z}.$$

Let $z = e^{i\theta}$, the solution becomes

$$\begin{aligned} f(e^{i\theta}) &= \frac{-1+e^{2i\theta}+\sqrt{e^{4i\theta}-2e^{2i\theta}+1+4|\alpha|^2e^{2i\theta}}}{2\overline{\alpha}e^{i\theta}} \\ &= \frac{2i\sin(\theta)+e^{-i\theta}\sqrt{e^{4i\theta}-2e^{2i\theta}+1+4|\alpha|^2e^{2i\theta}}}{2\overline{\alpha}} \\ &= \frac{i\sin(\theta)}{\overline{\alpha}}+\frac{e^{-i\theta}}{\sqrt{2\alpha}}\sqrt{e^{2i\theta}(\cos(2\theta)-(1-2|\alpha|^2))}. \end{aligned}$$

By multiplying $f(e^{i\theta})$ by $\overline{f(e^{i\theta})}$ on both sides, it is a straight-forward calculation to show that $|f| = 1$ for $\cos(2\theta) \geq (1-2|\alpha|^2)$ and $|f| < 1$ for $\cos(2\theta) < (1-2|\alpha|^2)$. Note that

$$\begin{aligned} \cos(2\theta) &< 1-2|\alpha|^2 \\ 2|\alpha|^2 &< 1-\cos(2\theta) \\ 2|\alpha|^2 &< 1-(1+2\sin^2(\theta)) \\ |\alpha| &< |\sin(\theta)|. \end{aligned}$$

This gives the support for the absolutely continuous part of the measure, which is $\{\theta : f(e^{i\theta}) < 1\}$. Below is the numerical approximation of $w(\theta)$. The red line is the a.c. density recovered from the real part of the Carathédory function. The *roots* of the grey line show where $|\alpha| = |\sin(\theta)|$. As predicted by the theory, the a.c. part is supported on $\{\theta : f(e^{i\theta}) < 1\}$. The reader is directed to Golinskii (1990) for another example of periodic Verblunsky coefficients.

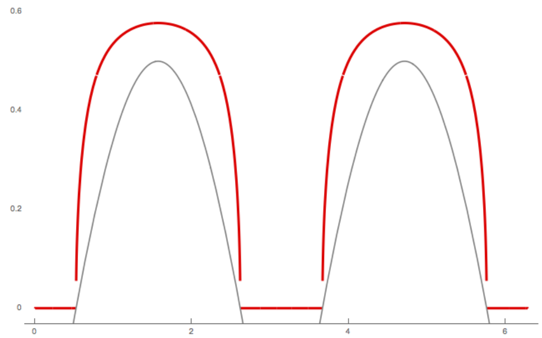

Figure 1: $\alpha = 0, \frac{1}{2}, 0, \frac{1}{2}, \ldots$

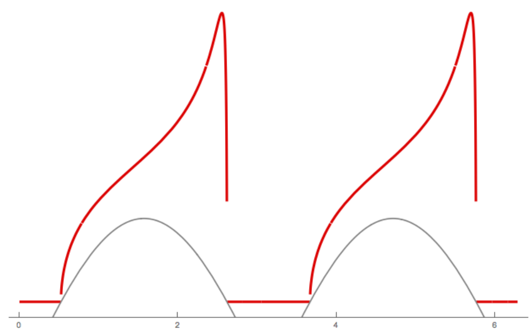

Figure 2: $\alpha = 0, \frac{i}{2}, 0, \frac{i}{2}, \ldots$

Example 4.12. $(\alpha, 0, \alpha, 0, \ldots)$ Begin with the Verblunsky coefficients as $\alpha, 0, \alpha, 0, \ldots$. That is,

$$\begin{aligned} \alpha_{2j} &= \alpha \\ \alpha_{2j+1} &= 0 \end{aligned}$$

In the previous example, we showed how Verblunsky coefficients similar to this form relate to the support of the measure. In this example, we show how the

Toeplitz determinants D_n can be computed explicitly. For even n,

$$\begin{aligned} D_{2n} &= \prod_{j=0}^{2n-1} (1-|\alpha_j|^2)^{2n-j} \\ &= (1-\alpha^2)^{2n}(1-\alpha^2)^{2n-2}\cdots(1-\alpha^2)^2 \\ &= (1-\alpha^2)^{\sum_{i=1}^{n} 2i} \\ &= (1-\alpha^2)^{n(n+1)}. \end{aligned}$$

Similarly for the odd n,

$$\begin{aligned} D_{2n+1} &= \prod_{j=0}^{2n}(1-\alpha^2)^{2n+1-j} \\ &= (1-\alpha^2)^{2n+1}(1-\alpha^2)^{(2n+1)-2}\cdots(1-\alpha^2)^{(2n+1)-2n} \\ &= (1-\alpha^2)^{n+1}(1-\alpha^2)^{n(n+1)} \\ &= (1-\alpha^2)^{n+1}D_{2n} \\ D_{2n-1} &= (1-\alpha^2)^{-n}D_{2n}. \end{aligned}$$

In summary, we have

$$\begin{aligned} D_{2n} &= (1-\alpha^2)^{n(n+1)} \\ D_{2n+1} &= (1-\alpha^2)^{(n+1)^2} \\ D_{2n-1} &= (1-\alpha^2)^{n^2}. \end{aligned}$$

Note that this indeed obeys the formula $1-\alpha_{2n}^2 = \frac{D_{2n+1}}{D_{2n}}\frac{D_{2n-1}}{D_{2n}}$.

Example 4.13. (Sine Carathédory Function) Consider $F(z) = 1+\sin(z)$. This is indeed a valid Carathédory function because $F(0) = 1$ and the real part is ≥ 0 as sine is bounded between -1 and 1. At $\partial\mathbb{D}$,

$$\begin{aligned} F(e^{i\theta}) &= 1+\sin(e^{i\theta}) \\ \mathrm{Re}F(e^{i\theta}) &= \frac{1}{2}(1+\sin e^{i\theta}+1+\sin e^{-i\theta}) \\ &= \frac{1}{2}(2+2\cosh(\sin\theta)\sin(\cos\theta)). \end{aligned}$$

Hence

$$w(\theta) \quad = \quad 1 + \cosh(\sin\theta)\sin(\cos\theta).$$

This gives the closed-form formula for the definite intergral:

$$\int_0^{2\pi} [1 + \cosh(\sin\theta)\sin(\cos\theta)]\frac{d\theta}{2\pi} = 1.$$

For $n \geq 1$, we also obtain the following closed-form formula. Interestingly, even the `NIntegrate` function in `Mathematica` fails in precision after the 15th moment.

$$\begin{aligned} c_n &= \int_0^{2\pi} \exp(-in\theta)[1 + \cosh(\sin\theta)\sin(\cos\theta)]\frac{d\theta}{2\pi} \\ &= \begin{cases} 0 & \text{if } n = \text{even} \\ \frac{1}{2}(-1)^{\frac{n-1}{2}}\frac{1}{n!} & \text{if } n = \text{odd}. \end{cases} \end{aligned}$$

Example 4.14. (Exponential Carathédory Function) Consider the measure μ whose moments are $c_n = \frac{1}{2}\frac{1}{n!}$. Because the moments are exponentially decreasing, we know there is no mutually singular part to the measure. By analyticity, the Carathédory function equals

$$F(z) \quad = \quad 1 + 2\sum_{n=1}^{\infty} c_n z^n = \exp(z).$$

On $\partial\mathbb{D}$, $z = e^{i\theta}$, so

$$\begin{aligned} F(e^{i\theta}) &= e^{\cos\theta + i\sin\theta} \\ &= e^{\cos\theta}e^{i\sin\theta} \\ &= e^{\cos\theta}[\cos(\sin\theta) + i\sin(\sin\theta)]. \end{aligned}$$

To get the density $w(\theta)$, we simply take the real part

$$w(\theta) = \mathrm{Re}F(e^{i\theta}) = e^{\cos\theta}\cos(\sin\theta).$$

Notice that this is indeed a valid probability density because $\sin\theta$ ranges from

-1 to 1, and cosine is positive between $-\frac{\pi}{2} < -1 < 1 < \frac{\pi}{2}$. Again, we have formulas giving explicit values for the following definite integrals:

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta) \frac{d\theta}{2\pi} = 1.$$

For $n \geq 1$

$$\int_0^{2\pi} e^{-in\theta} e^{\cos\theta} \cos(\sin\theta) \frac{d\theta}{2\pi} = \frac{1}{2}\frac{1}{n!}.$$

5 Random Bernoulli Verblunsky Coefficients

5.1 Introduction

Let the Verblunsky coefficients $\{\alpha_i\}$ be i.i.d Bernoulli random variables which takes on values a or b with equal probability. Assume throughout this section that $a, b \in \mathbb{D}$ and $a \neq b \neq 0$, as the zero case is uninteresting. Given a sequence of random Verblunsky coefficients, we are interested in the distribution of zeros for Φ_n (also random). The zeros of Φ_n are of intrinsic interest because they are the eigenvalues of CMV matrices. The analog on the real line is that the zeros of the orthogonal polynomials are the energy eigenvalues for quantum operators.

Because each α_i is drawn from a finite sample space, for fixed n, the list $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ has 2^n possible realizations. The unique 2^n lists of Verblunsky coefficients also produce 2^n monic OP's (Φ_n) of degree n. The natural question to ask here is whether two different lists of Verblunsky coefficients can produce the same Φ_n. The answer is no, as can be seen from Theorem 3.4. Starting with Φ_n, we can recover all the unique Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. Here we provide an alternative proof for the uniqueness of Φ_n.

Theorem 5.1. *(Uniqueness of the monic OP's) Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\beta_0, \beta_1, \ldots, \beta_n$ be two lists of Verblunsky coefficients that produce the same Φ_{n+1}. Then $\alpha_i = \beta_i$ for $i = 0, 1, \ldots, n$. That is, the monic OP's uniquely determine all the previous Verblunsky coefficients.*

Proof. Note $\alpha_n = \beta_n$ because they are both equal to $-\overline{\Phi_{n+1}(0)}$. Let $\Phi_n(\alpha)$ and $\Phi_n(\beta)$ denote the degree n monic OP's produced from the α_i's and β_i's respectively. It is enough to show that $\Phi_n(\alpha) = \Phi_n(\beta)$ because that implies $\alpha_{n-1} = \beta_{n-1}$ and the procedure can be repeated iteratively. Denote $\alpha_n = \beta_n = \overline{\gamma}$. Let

$$\begin{aligned}
\Phi_n(\alpha) &= z^n + \sum_{j=0}^{n-1} A_j z^j \\
\Phi_n(\beta) &= z^n + \sum_{j=0}^{n-1} B_j z^j.
\end{aligned}$$

Consequently, the reverse polynomials are

$$\begin{aligned}\Phi_n(\alpha)^* &= 1+\sum_{j=1}^{n}\overline{A_{n-j}}z^j\\ \Phi_n(\beta)^* &= 1+\sum_{j=1}^{n}\overline{B_{n-j}}z^j.\end{aligned}$$

From the Szegö recurrence (3.3), we have

$$\begin{aligned}\Phi_{n+1} &= z(z^n+\sum_{j=0}^{n-1}A_jz^j)-\gamma(1+\sum_{j=1}^{n}\overline{A_{n-j}}z^j)\\ &= z^{n+1}+\sum_{j=1}^{n}(A_{j-1}-\gamma\overline{A_{n-j}})z^j-\gamma\\ &= z^{n+1}+\sum_{j=1}^{n}(B_{j-1}-\gamma\overline{B_{n-j}})z^j-\gamma.\end{aligned}$$

Equating the coefficients, we have the system of equations

$$\begin{aligned}A_0-\gamma\overline{A_{n-1}} &= B_0-\gamma\overline{B_{n-1}}\\ A_1-\gamma\overline{A_{n-2}} &= B_1-\gamma\overline{B_{n-2}}\\ &\vdots\\ A_{n-2}-\gamma\overline{A_1} &= B_{n-2}-\gamma\overline{B_1}\\ A_{n-1}-\gamma\overline{A_0} &= B_{n-1}-\gamma\overline{B_0}.\end{aligned}$$

We need to show $A_i = B_i$ for all i, which implies $\Phi_n(\alpha) = \Phi_n(\beta)$ and finishes the proof. Note that if $\gamma = 0$, it is trivially true that $A_i = B_i$ for all i, so assume that $\gamma \neq 0$. To solve this system of equations, pair the first equation with the last equation, the second equation with the second last equation, and so on. For n even, there will be $\frac{n}{2}$ pairs of equations with no overlapping indices (e.g. 0 and $n-1$). For n odd, there will be an equation with overlapping index in the middle. First, consider the overlapping index case. We solve the equation

$$A_i-\gamma\overline{A_i} = B_i-\gamma\overline{B_i}.$$

Denote $\text{Re}(A_i) = a_i$ and $\text{Im}(A_i) = \widetilde{a_i}$ and similary for B_i. The equation above becomes

$$a_i + i\widetilde{a_i} - \gamma(a_i - i\widetilde{a_i}) \quad = \quad b_i + i\widetilde{b_i} - \gamma(b_i - i\widetilde{b_i}).$$

Equating the real and imaginary parts, we have

$$(1-\gamma)a_i = (1-\gamma)b_i$$
$$(1+\gamma)\widetilde{a_i} = (1+\gamma)\widetilde{b_i}.$$

Since γ is a Verblunsky coefficient, $|\gamma| < 1$, so $(1-\gamma) \neq 0$ and $(1+\gamma) \neq 0$. Therefore, $a_i = b_i$ and $\widetilde{a_i} = \widetilde{b_i}$, which completes the proof for the overlapping index. Finally, consider a pair of equations without overlapping indices. Without loss of generality, we look at the pair 0 and $n-1$, as the algebra is exactly the same for other pairs:

$$A_0 - \gamma\overline{A_{n-1}} \quad = \quad B_0 - \gamma\overline{B_{n-1}}$$
$$A_{n-1} - \gamma\overline{A_0} \quad = \quad B_{n-1} - \gamma\overline{B_0}.$$

Again, denoting $\text{Re}(A_i) = a_i$ and $\text{Im}(A_i) = \widetilde{a_i}$ and similary for B_i, we have

$$a_0 + i\widetilde{a_0} - \gamma(a_{n-1} - i\widetilde{a}_{n-1}) \quad = \quad b_0 + i\widetilde{b_0} - \gamma(b_{n-1} - i\widetilde{b}_{n-1}) \tag{5.1}$$
$$a_{n-1} + i\widetilde{a}_{n-1} - \gamma(a_0 - i\widetilde{a_0}) \quad = \quad b_{n-1} + i\widetilde{b}_{n-1} - \gamma(b_0 - i\widetilde{b_0}). \tag{5.2}$$

Equating the real parts for (5.1) and (5.2), we have

$$a_0 - \gamma a_{n-1} \quad = \quad b_0 - \gamma b_{n-1}$$
$$a_{n-1} - \gamma a_0 \quad = \quad b_0 - \gamma b_0.$$

Multiplying the second equation by $\frac{1}{\gamma}$ and adding it to the first equation gives us

$$\left(\frac{1}{\gamma} - \gamma\right) a_{n-1} \quad = \quad \left(\frac{1}{\gamma} - \gamma\right) b_{n-1}.$$

Since $|\gamma| < 1$, $\gamma \neq \pm 1$. Thus, $a_{n-1} = b_{n-1}$ and $a_0 = b_0$. We can similarly equate the imaginary parts for (5.1) and (5.2) and obtain

$$\left(\gamma - \frac{1}{\gamma}\right)\widetilde{a}_{n-1} \quad = \quad \left(\gamma - \frac{1}{\gamma}\right)\widetilde{b}_{n-1},$$

which shows that the imaginary parts are equal as well. This shows that $A_i = B_i$ for every i, which completes the proof. □

5.2 The Zeros

Theorem 5.1 establishes that if α_i are i.i.d Bernoulli random variables which take on values a or b with equal probability, then for any fixed level n, the list of possible monic OP's $\{\Phi_{n,i}(z)\}_{i=1}^{i=2^n}$ contains 2^n unique degree n monic polynomials. Because we are interested in the property of zeros, we first need some preliminary results on the properties of those zeros.

Theorem 5.2. *(Zeros Theorem) All the zeros of $\Phi_n(z)$ lie in the open unit disk $\mathbb{D}$ for non-trivial probability measures on $\partial\mathbb{D}$.*

Proof. There are several proofs for this theorem. The most straight-forward proof only relies on the fact that the Φ_n's obey the recurrence relationships (3.3) and that $|\alpha_i| < 1$. For the proof, see Simon (2005*b*). □

Corollary 5.3. *(Zeros of $\Phi_n^*(z)$) All the zeros of $\Phi_n^*(z)$ lie strictly outside the closed unit disk. That is, if z_0 is a zero of $\Phi_n^*(z)$, then $z_0 \in \mathbb{C}\backslash(\mathbb{D} \cup \partial\mathbb{D})$.*

Proof. The proof follows from the definition of the reverse polynomial:

$$\Phi_n^*(z) \quad = \quad z^n \overline{\Phi_n\left(\frac{1}{\overline{z}}\right)}.$$

Since the zeros of Φ_n lie strictly within the open unit disk, $1/\overline{z}$ forces the zeros of $\Phi_n^*(z)$ to lie strictly outside the closed unit disk. In fact, if $z_1, z_2, \ldots, z_n$ are the zeros of $\Phi_n(z)$, then $\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}$ are the zeros of $\Phi_n^*(z)$. □

Corollary 5.4. *For all $z \in \mathbb{D} \cup \partial\mathbb{D}$, we have*

$$|\Phi_n(z)| \leq |\Phi_n^*(z)|$$

for $n > 0$. If $z \in \mathbb{D}$, then the inequality is strict.

Proof. The reader is directed to Simon (2005*b*) for the proof. □

Theorem 5.5. *Let $n \geq 2$. Suppose $\alpha_{n-1} \neq 0$. Then $\Phi_n(z)$ has no zeros in the disk*

$$\{z : |z| \leq |\alpha_{n-1}|\}.$$

Remark 5.6. This theorem provides a simple lower bound for the size of the zeros. However, it is not a very good one for our discrete random Verblunsky coefficients case. This is because the bound $|\alpha| = \min(|a|, |b|)$ will be independent of n. Since as $n \to \infty$, the zeros of Φ_n approach $\partial\mathbb{D}$, the bound becomes increasingly worse as n increases.

Proof. Let z_0 be a zero of $\Phi_n(z)$. Plugging this into the Szegö recurrence (3.3),

$$0 = z_0\Phi_{n-1}(z_0) - \overline{\alpha_{n-1}}\Phi_n^*(z_0).$$

Taking norms, we obtain

$$|z_0|\frac{|\Phi_{n-1}(z_0)|}{|\Phi_{n-1}^*(z_0)|} = |\alpha_{n-1}|.$$

By Corollary 5.4, we must have $|z_0| > |\alpha_{n-1}|$. □

From Theorem 5.2, we know that every Φ_n has n zeros that all lie within the open unit disk $\mathbb{D}$. For fixed n, the density of zeros can be defined as the probability of finding a particular zero (there are $n * 2^n$ zeros in total for every n). If all the zeros are distinct, then every zero occurs with equal probability $\frac{1}{n2^n}$. Under what conditions are there no common zeros amongst $\{\Phi_{n,i}\}$? The next few theorems provide a partial answer to that question.

Theorem 5.7. *Conditional on the realization of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, the two possible Φ_{n+1} (generated from $\alpha_n = a$ or $\alpha_n = b$) have no common zeros.*

Proof. Consider the Szegö recurrence, separately for a and b, $a \neq b$. Denote $\Phi_{n+1}(a)$ and $\Phi_{n+1}(b)$ as the monic OPs produced from a and b respectively.

Then

$$\begin{aligned}\Phi_{n+1}(a) &= z\Phi_n - \overline{a}\Phi^* \\ \Phi_{n+1}(b) &= z\Phi_n - \overline{b}\Phi^*.\end{aligned}$$

Suppose for contradiction that ζ is a common zero of $\Phi_{n+1}(a)$ and $\Phi_{n+1}(b)$. Then we have

$$\begin{aligned}0 &= \zeta\Phi_n(\zeta) - \overline{a}\Phi_n^*(\zeta) \\ 0 &= \zeta\Phi_n(\zeta) - \overline{b}\Phi_n^*(\zeta),\end{aligned}$$

which gives us

$$\overline{(a-b)}\Phi_n^*(\zeta) = 0.$$

Since ζ is a zero of a monic OP, it must lie strictly within $\mathbb{D}$. The reverse monic OP Φ_n^* has all of its zeros strictly outside $\mathbb{D}$, hence $\Phi_n^*(\zeta) \neq 0$. We conclude that $a = b$, which is a contradiction. Therefore, $\Phi_{n+1}(a)$ and $\Phi_{n+1}(b)$ must not have common zeros. □

Lemma 5.8. *If $\alpha_n \neq 0$, then Φ_n and Φ_{n+1} have no common zeros.*

Proof. Suppose for contradiction that ζ is a common zero of Φ_n and Φ_{n+1}. By the Szegö recurrence,

$$\begin{aligned}\Phi_{n+1}(\zeta) &= \zeta\Phi_n(\zeta) - \overline{\alpha}_n\Phi_n^*(\zeta) \\ 0 &= -\overline{\alpha}_n\Phi_n^*(\zeta).\end{aligned}$$

Note that ζ is not a zero of Φ_n^* because $\zeta \in \mathbb{D}$ and the zeros of Φ_n^* lie outside $\mathbb{D}$. Therefore, $\alpha_n = 0$, which is a contradiction. □

Theorem 5.9. *Conditional on the realization of $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$, any realization of Φ_{n+1} has unique zeros.*

Proof. Since α_i is a Bernoulli random variable, there are four possible realizations of α_{n-1} and α_n: $(a,a), (b,b), (a,b)$ and (b,a). Each pair creates a different

Φ_{n+1}. We show that any two Φ_{n+1} created will have no common zeros. There are three cases to consider.

Case 1: If the α_{n-1}'s are equal and α_n's are different, then by Theorem 5.7, the two possible Φ_{n+1} will have no common zeros.

Case 2: Consider the case in which the α_n's is the same but α_{n-1} is different. That is, consider $\Phi_{n+1}(a,a)$ and $\Phi_{n+1}(b,a)$. Suppose for contradiction that ζ is a common zero. Then by (3.3),

$$\begin{aligned} 0 &= \zeta(\Phi_n(\zeta;a)) - \overline{a}\Phi_n^*(\zeta;a) \\ 0 &= \zeta(\Phi_n(\zeta;b)) - \overline{a}\Phi_n^*(\zeta;b), \end{aligned}$$

where $\Phi_n(;a)$ and $\Phi_n(;b)$ denote Φ_n produced from $\alpha_{n-1} = a$ and b respectively. Using (3.3), (3.4) for Φ_n and noting that Φ_{n-1} is equal for both (a,a) and (b,a),

$$\begin{aligned} 0 &= \zeta(\zeta\Phi_{n-1} - \overline{a}\Phi_{n-1}^*) - \overline{a}(\Phi_{n-1}^* - \overline{a}\zeta\Phi_{n-1}) \\ 0 &= \zeta(\zeta\Phi_{n-1} - \overline{b}\Phi_{n-1}^*) - \overline{a}(\Phi_{n-1}^* - \overline{b}\zeta\Phi_{n-1}). \end{aligned}$$

Combining the two equations, we have

$$-\zeta\overline{a}\Phi_{n-1}^* + \overline{a}(\overline{a}\zeta\Phi_{n-1}) = -\zeta\overline{b}\Phi_{n-1}^* + \overline{b}(\overline{a}\zeta\Phi_{n-1}).$$

Note that ζ is not a root of Φ_{n-1}^* because ζ is a root for Φ_{n+1} so $\zeta \in \mathbb{D}$, and the roots of Φ_{n-1}^* lie strictly outside of $\mathbb{D}$. Furthermore, $\zeta \neq 0$ because $a, b \neq 0$, so any monic OP generated from the Bernoulli Verblunsky coefficients cannot have 0 as a root. Therefore, ζ is not a root of Φ_{n-1} because if it were, we reach a contradiction that $a = b$. Now, divide both sides by ζ and rearrange:

$$\begin{aligned} -\overline{a}\Phi_{n-1}^* + \overline{a}^2\Phi_{n-1} &= -\overline{b}\Phi_{n-1}^* + \overline{a}\overline{b}\Phi_{n-1} \\ -\overline{a}(\Phi_{n-1}^* + \overline{a}\Phi_{n-1}) &= -\overline{b}(\Phi_{n-1}^* + \overline{a}\Phi_{n-1}). \end{aligned}$$

If we show that $\Phi_{n-1}^*(\zeta) \neq \overline{a}\Phi_{n-1}(\zeta)$, then we are done with this case, because this leads to $a = b$, which is a contradiction. Suppose for contradiction that this were the case. Then we have

$$\overline{a} \;=\; \frac{\Phi^*_{n-1}(\zeta)}{\Phi_{n-1}(\zeta)}.$$

Taking norms on both sides,

$$|a| \;=\; \frac{|\Phi^*_{n-1}(\zeta)|}{|\Phi_{n-1}(\zeta)|}.$$

This is a contradiction because $|a| < 1$ as it is a Verblunsky coefficient for a non-trivial probability measure on $\partial\mathbb{D}$ while $|\Phi^*_{n-1}(z)| > |\Phi_{n-1}(z)|$ for all $z \in \mathbb{D}$.

Case 3: Consider the case in which both α_{n-1} and α_n are different. That is, consider two Φ_{n+1}'s created by (a, a) and (b, b). Suppose for contradiction that they have a common zero ζ. Using (3.3) and (3.4) two times, we have

$$\begin{aligned} 0 &= \zeta(\zeta\Phi_{n-1} - \overline{a}\Phi^*_{n-1}) - \overline{a}(\Phi^*_{n-1} - \overline{a}\zeta\Phi_{n-1}) \\ 0 &= \zeta(\zeta\Phi_{n-1} - \overline{b}\Phi^*_{n-1}) - \overline{b}(\Phi^*_{n-1} - \overline{b}\zeta\Phi_{n-1}); \end{aligned}$$

simplifying, we have

$$\begin{aligned} 0 &= -\zeta\overline{a}\Phi^*_{n-1} - \overline{a}\Phi^*_{n-1} + \overline{a}(\overline{a}\zeta\Phi_{n-1}) \\ 0 &= -\zeta\overline{b}\Phi^*_{n-1} - \overline{b}\Phi^*_{n-1} + \overline{b}(\overline{b}\zeta\Phi_{n-1}). \end{aligned}$$

Factoring out $\overline{a}$ and $\overline{b}$,

$$\begin{aligned} 0 &= \overline{a}(-\zeta\Phi^*_{n-1} - \Phi^*_{n-1} + \overline{a}\zeta\Phi_{n-1}) \\ 0 &= \overline{b}(-\zeta\Phi^*_{n-1} - \Phi^*_{n-1} + \overline{b}\zeta\Phi_{n-1}). \end{aligned}$$

These two equations show that ζ is not a root of Φ_{n-1} because if it were, then $a = b = 0$, which contradicts our initial assumption. Next, we divide the equations by $\overline{a}$ and $\overline{b}$ respectively:

$$\begin{aligned} 0 &= -\zeta\Phi^*_{n-1} - \Phi^*_{n-1} + \overline{a}\zeta\Phi_{n-1} \\ 0 &= -\zeta\Phi^*_{n-1} - \Phi^*_{n-1} + \overline{b}\zeta\Phi_{n-1}. \end{aligned}$$

Combining the two equations and canceling terms, we have

$$(\overline{a-b})(\zeta\Phi_{n-1}(\zeta)) \;=\; 0.$$

Since ζ is not a root of Φ_{n-1} and $\zeta \neq 0$, we get that $a = b$, which is a contradiction. □

Theorem 5.10. *Let Φ_{n+1} and $\widetilde{\Phi}_{n+1}$ be two degree $n+1$ monic OP which have the same constant term (i.e. they are produced by the same α_n). If z_0 is a common zero for Φ_{n+1} and $\widetilde{\Phi}_{n+1}$, then*

$$\frac{\widetilde{\Phi}_n^*(z_0)}{\widetilde{\Phi}_n(z_0)} \;=\; \frac{\Phi_n^*(z_0)}{\Phi_n(z_0)}. \tag{5.3}$$

Proof. First, note that (5.3) makes sense because z_0 is not a zero of $\widetilde{\Phi}_n$ or Φ_n by Lemma 5.8. Also note that z_0 is not a zero of $\widetilde{\Phi}_n^*$ or Φ_n^* because the zeros of the reverse polynomials lie strictly outside of $\mathbb{D}$. Since z_0 is a zero of Φ_{n+1} and $\widetilde{\Phi}_{n+1}$, evaluate (3.3) for both polynomials:

$$0 \;=\; z_0\Phi_n(z_0) - \overline{\alpha}_n\Phi_{n-1}^*(z_0) \tag{5.4}$$

$$0 \;=\; z_0\widetilde{\Phi}_n(z_0) - \overline{\alpha}_n\widetilde{\Phi}_{n-1}^*(z_0). \tag{5.5}$$

Solving for z_0 from (5.4), we get

$$z_0 \;=\; \frac{\overline{\alpha_n}\Phi_n^*(z_0)}{\Phi_n(z_0)}.$$

Plugging this into (5.5), we get

$$1 = \left(\frac{\Phi_n^*(z_0)}{\Phi_n(z_0)}\right)\left(\frac{\widetilde{\Phi}_n(z_0)}{\widetilde{\Phi}_n^*(z_0)}\right),$$

which shows (5.3). □

5.3 The Case $a = -b$ and Transformations of the Measure

As before, let the list of Verblunsky coefficients be i.i.d. Bernoulli random variables which take on values a and b with equal probability. In this section, we further restrict that $a = -b$.

Let the Verblunsky coefficients take on values $\gamma, -\gamma \in \mathbb{D}$. For any fixed n, this gives rise to 2^{n+1} possible realizations of $\alpha_0, \alpha_1, \ldots, \alpha_n$ and hence 2^{n+1} possible realizations of measure $\mu(\alpha_0, \alpha_1, \ldots, \alpha_n)$. As we will see, we can identify different realizations of measures which are transformations of each other. Consequently, this gives rise to symmetries in the distribution of zeros and moments. First, we need some results on the transformation of the measure μ. For the detailed derivations, the reader is directed to Simon (2005*b*).

Let $\omega = e^{i\theta_0}$ be a rotation of the measure $d\mu(\theta)$ to $d\nu = d\mu(\theta - \theta_0)$ (i.e. the measure is rotated counter-clockwise by a3ngle θ_0). Let Z_n be the set of zeros of Φ_n. Then, the following relations hold:

$$\begin{aligned} c_n(d\nu) &= \omega^{-n} c_n(d\mu) \\ \alpha_n(d\nu) &= \omega^{-(n+1)} \alpha_n(d\mu) \\ Z_n(d\nu) &= e^{i\theta_0} Z_n(d\mu). \end{aligned}$$

If we consider the special case where $\theta_0 = \pi$, so that $\omega = e^{i\pi} = -1$. Then we have that

$$\begin{aligned} c_n(d\nu) &= (-1)^n c_n(d\mu) \\ \alpha_n(d\nu) &= (-1)^{(n+1)} \alpha_n(d\mu) \\ Z_n(d\nu) &= (-1) Z_n(d\mu). \end{aligned}$$

This tells us that if the measure is rotated by an angle of π, then the even moments stay the same while the odd moments switch sign. The odd Verblunsky coefficients stay the same while the even moments switch sign. The zeros are rotated by π. This will help us identify different realizations of $\alpha_0, \alpha_1, \ldots, \alpha_n$ which are rotations of the measure by π.

Next, consider when the measure is reflected about the real axis. That is, consider the transformation of the measure from $d\mu(\theta)$ to $d\nu = d\mu(-\theta)$. A simple calculation will give us that

$$\begin{aligned} c_n(d\nu) &= \overline{c_n(d\mu)} \\ \alpha_n(d\nu) &= \overline{\alpha_n(d\mu)} \\ Z_n(d\nu) &= \overline{Z_n(d\mu)}. \end{aligned}$$

This result combined with the result for rotation by π also allows us to calculate what happens when the measure is reflected about the imaginary axis (i.e. $d\mu(\theta)$ to $d\nu = d\mu(\pi - \theta)$), because that is a rotation by π followed by a reflection about the real axis. Knowing how the different objects relate under transformations of μ, we may now characterize some properties of when $a = -b$.

In the following theorems, let the Verblunsky coefficients be i.i.d Bernoulli random variables which take on values γ and $-\gamma$ with $|\gamma| < 1$. For any fixed $(n-1)$, we can talk about the *distribution of* c_n as the set of the 2^n possible c_n's from different realizations of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. This is because $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ uniquely determine $c_1, \ldots, c_n$ through (3.38). Similarly, we can talk about the distribution of Z_n as the set of $n2^n$ possible zeros from the 2^n possible Φ_n's.

Theorem 5.11. *Let $\gamma \in \mathbb{D}$. For every even n, the distribution of c_n takes on at most 2^{n-1} values. Specifically, every value in the distribution of c_n appears at least twice.*

Proof. The proof follows from noting that for even n, the realization $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and $(-\alpha_0, \alpha_1, -\alpha_2, \ldots, -\alpha_{n-1})$ produce the same moments. This is because under a rotation of the measure by π, the even moments stay fixed. Therefore, every value in the distribution of c_n appears at least twice, giving us at most $2^n/2 = 2^{n-1}$ possible values. □

Theorem 5.12. *If $\gamma \in \mathbb{R}$, then for every odd n, the distribution of c_n is symmetric around* 0.

Proof. In this case, because the Verblunsky coefficients are real, the moments are also real, so it makes sense to talk about the distribution of moments on $\mathbb{R}$. For the proof, take any realization of Verblunsky coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$. We know that $(-\alpha_0, \alpha_1, -\alpha_2, \ldots, -\alpha_{n-1})$ is also a valid realization. These two realizations produce c_n and $-c_n$, as they correspond to rotations of the measures by π. Since we can pair every possible realization of Verblunsky coefficients with a rotation by π, it follows that for every moment c_n, $-c_n$ is also in the distribution. □

Theorem 5.13. *If $\gamma \in i\mathbb{R}$, then:*

(i) For every n, the distribution of c_n is symmetric about the real axis (i.e. the moments come in complex conjugate pairs in the distribution).

(ii) The distribution of every odd n is symmetric around the origin.

Proof. To see (i), note that for any realization $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, $(\overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}})$ is also a valid realization, because γ is purely imaginary. This implies that for every possible value in the distribution of c_n, $\bar{c}_n$ is also in the distribution, because we can obtain it from the realization $(\overline{\alpha_0}, \overline{\alpha_1}, \ldots, \overline{\alpha_{n-1}})$. This shows the first statement.

To prove (ii), it is enough to show that for odd n, $-\overline{c_n}$ is also in the distribution of c_n. This is because for any $c_n \in \mathbb{C}$,

$$\begin{aligned} -\overline{c_n} &= -(\mathrm{Re}(c_n) - \mathrm{Im}(c_n)) \\ &= -\mathrm{Re}(c_n) + \mathrm{Im}(c_n)) \end{aligned}$$

so that we indeed have a reflection across the imaginary axis. Note that for odd n and every realization $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, the realization $(\alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \ldots, \alpha_{n-1})$ is a reflection across the imaginary axis, which maps the odd moments c_n to $-\bar{c}_n$. A reflection across the real axis followed by a rotation by π maps the Verblunsky coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ to $(\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1})$ and then to $(-\overline{\alpha}_0, \overline{\alpha}_1, -\overline{\alpha}_2, \ldots, \overline{\alpha}_{n-1})$. But since every α_i is purely imaginary, we have that $(-\overline{\alpha}_0, \overline{\alpha}_1, -\overline{\alpha}_2, \ldots, \overline{\alpha}_{n-1}) = (\alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \ldots, \alpha_{n-1})$. This composition of transformations gives us exactly a reflection across the imaginary axis. Since we have shown that the distribution of the odd c_n's is symmetric about both the real axis and the imaginary axis, we have completed the proof of (ii). □

The proof of the previous theorem essentially identified four different realizations of Verblunsky coefficients which correspond to four different measures that are transformations of each other (rotation by π, reflection across the imaginary axis, and reflection across the real axis). Identifying the different realizations that are transformations of each other also gives information on the distribution of zeros, because the zeros are transformed the same way as the measures are transformed. We therefore have the next two corollaries.

Corollary 5.14. *Let $\gamma \in i\mathbb{R}$. There are four possible realizations of $\Phi_2(z)$ that come from $(\alpha_0, \alpha_1) = (\gamma, \gamma)$, $(-\gamma, \gamma)$, $(\gamma, -\gamma)$, and $(-\gamma, -\gamma)$. Then the distribution of the zeros of Φ_2 ($4 \times 2^2 = 16$ possible values) will have no zeros that are purely imaginary or purely real. Moreover, if ζ is a zero of any Φ_2, then $-\zeta, \overline{\zeta}$, and $-\overline{\zeta}$ are not zeros of any Φ_2.*

Proof. We know from the proof of Theorem 5.13 that these four possible realizations are transformations of each other. Namely, they correspond to when the measure is rotated by π and reflected across the imaginary and real axes. Consequently, the zeros of the four Φ_2's are also transformed accordingly. By Theorem 5.9, none of them have common zeros. It follows that if ζ is a zero of any Φ_2, $-\zeta, \overline{\zeta}$, and $-\overline{\zeta}$ cannot be the zeros of any Φ_2 or they would have common zeros. □

Corollary 5.15. *For any $\gamma \in \mathbb{D}$ and any $n \geq 1$, no realization of Φ_n is an even function.*

Proof. Suppose for contradiction that there exists a realization of Φ_n that is an even function. Let this come from the realization $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ and denote the corresponding measure as μ. Denote the monic OP obtained from rotating the measure by π as $\widetilde{\Phi}_n$. Since Φ_n is even, for every zero z_i of Φ_n, $-z_i$ is also a zero of Φ_n. It then follows that $\Phi_n = \widetilde{\Phi}_n$ because the zeros remain unchanged under the rotation by π. Since $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ and $(-\alpha_0, \alpha_1, -\alpha_2, \ldots, \alpha_{n-1})$ corresponds to rotation by π, we have identified two *different* realizations of Φ_n which are identical, contradicting Theorem 5.1. Therefore, no Φ_n can be an even function. □

In fact, very generally speaking, if Φ_n is an even function, we can say something about the list of Verblunsky coefficients producing it and the lower order monic OP's. This is summarized in the following lemma.

Lemma 5.16. *For fixed n, if Φ_n contains only even powers, then for $0 \leq i \leq n$:*

(i) $\alpha_i = 0$ *for even i,*

(ii) If i is odd, Φ_i is an odd function and Φ_i^* is an even function, and

(iii) If i is even, Φ_i is an even function and Φ_i^* is an even function.

(Note that by the Inverse Szegö Theorem (3.4)*, Φ_n uniquely determines every Φ_i for $0 \leq i \leq n$ and consequently the list $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, so this lemma makes sense. This result does not rely on the Verblunsky coefficients being Bernoulli distributed.)*

Proof. Suppose that Φ_n have only even powers, so it is an even function. Let Φ_n be produced by the list $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ via the Szegö recurrence (3.3). Since Φ_n is an even function, it follows that the zeros of Φ_n come in plus minus pairs (i.e. if ζ is a zero of Φ_n then $-\zeta$ is also a zero). It follows that $(-\alpha_0, \alpha_1, -\alpha_2, \ldots, -\alpha_{n-2}, \alpha_{n-1})$ must produce the same Φ_n because the zeros are invariant under rotation of the measure by π. (Strictly speaking, we don't have a measure yet because we only have n Verblunsky coefficients. However, we can consider the "natural" measure by taking $\alpha_j = 0$ for all $j \geq n$.)

By Theorem 5.1, the two lists of Verblunsky coefficients must be equal to each other. This implies that $\alpha_j = 0$ for all even j, $0 \leq j \leq n$, which proves (i). We now prove (ii) and (iii) by induction. For $n = 1$, we have

$$\begin{aligned}\Phi_1 &= z\Phi_0 - \alpha_0\Phi_0^* \\ &= z,\end{aligned}$$

so that

$$\Phi_1^* = 1.$$

For $n = 2$, we have

$$\begin{aligned}\Phi_2 &= z\Phi_1 - \overline{\alpha}_1\Phi_1^* \\ &= z(z) - \overline{\alpha_1} \\ &= z^2 - \overline{\alpha_1},\end{aligned}$$

so that

$$\Phi_2^* = 1 - \alpha_1 z^2.$$

Since the base cases are true, it is enough to prove that this is true for Φ_{n+1} when $(n+1)$ is even and when $(n+1)$ is odd. We simply evaluate Φ_{n+1} and $\Phi^*_{n+1}(z)$ using (3.3) and (3.4). When $(n+1)$ is even, n is odd, so we have

$$\begin{aligned}
\Phi_{n+1}(-z) &= -z\Phi_n(-z) - \overline{\alpha}_n\Phi^*_n(-z) \\
&= z\Phi_n(z) - \overline{\alpha}_n\Phi^*_n(z) \\
&= \Phi_{n+1}(z) \\
\Phi^*_{n+1}(-z) &= \Phi^*_n(-z) - \alpha_n(-z)\Phi_n(-z) \\
&= \Phi^*_n(z) - \alpha_n z\Phi_n(z) \\
&= \Phi^*_{n+1}(z).
\end{aligned}$$

When $(n+1)$ is odd, n is even. Also remember that in this case, $\alpha_n = 0$. We have that

$$\begin{aligned}
\Phi_{n+1}(-z) &= -z\Phi_n(-z) - \overline{\alpha}_n\Phi^*_n(-z) \\
&= -z\Phi_n(z) \\
&= -\Phi_{n+1}(z) \\
\Phi^*_{n+1}(-z) &= \Phi^*_n(-z) - \alpha_n(-z)\Phi_n(-z) \\
&= \Phi^*_n(z) \\
&= \Phi^*_{n+1}(z).
\end{aligned}$$

Thus, we have shown by induction that (i) and (ii) are indeed true. □

Finally, since the zeros are transformed the same way as the measure μ is transformed, we have the following theorems characterizing the distribution of zeros for the case $\gamma, -\gamma$. Let Z_n be the zeros of one realization of $\Phi_n(z)$. We call the $n \times 2^n$ possible zeros from the 2^n possible realizations of $\Phi_n(z)$ the *distribution of* Z_n.

Theorem 5.17. *Let $\gamma \in i\mathbb{R}$ or $\gamma \in \mathbb{R}$. The distribution of Z_n is symmetric about the origin. That is, if ζ is in the distribution of Z_n, then $-\zeta, \overline{\zeta}$ and $-\overline{\zeta}$ are also in the distribution of Z_n.*

Proof. For the case $\gamma \in i\mathbb{R}$, we can associate the reflection across the real axis by the map from $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ to $\overline{(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})} = (-\alpha_0, -\alpha_1, \ldots, -\alpha_{n-1})$.

The reflection across the imaginary axis comes from a rotation by π followed by a reflection across the real axis, as shown in the proof of Theorem 5.13. These three transformations of μ, identified with 4 distinct possible realizations of $\Phi_n(z)$, give us the zeros $\zeta, -\zeta, \overline{\zeta}$ and $-\overline{\zeta}$. For the case $\gamma \in \mathbb{R}$, note that the zeros are symmetric across the real axis. This is because the coefficients of $\Phi_n(z)$ are real, so the zeros come in complex conjugate pairs. Rotation by π can be obtained via the map $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \to (-\alpha_0, \alpha_1, -\alpha_2, \ldots, (-1)^n \alpha_{n-1})$. This combined with the symmetry about the real axis gives us symmetry across the imaginary axis. Hence, we can also identify $\zeta, -\zeta, \overline{\zeta}$ and $-\overline{\zeta}$ in this case. $\square$

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6 Appendix A: The Three Main Convergence Theorems

Theorem 6.1. *(Monotone Convergence)*

Let $\{f_n\}$ be a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j and $f = \lim_{n\to\infty} f_n$, then $\int f = \lim_{n\to\infty} \int f_n$. In words, given a sequence of monotonically increasing, measurable functions that converge to f in L^+, we may interchange taking the limit and integration.

Proof. First, $\{\int f_n\}$ is an increasing sequence in $[0, \infty]$, so its limit exists. To show that $\lim \int f_n \leq \int f$, observe that $\int f_n \leq \int f$ for all n. To show that $\lim \int f_n \geq \int f$, fix $\alpha \in (0, 1)$. Now, define:

$$E_n = \{x : f_n(x) \geq \alpha\varphi(x)\}$$

where $\varphi(x)$ is a simple function such that $0 \leq \varphi \leq f$. It is not hard to see that E_n is a sequence of measurable sets whose union is X. It follows that:

$$\int f_n \geq \int_{E_n} f_n \geq \alpha \int_{E_n} \varphi$$

Since the integral, treated as a map from $E \in \mathbb{M}$ to $[0, \infty]$, is a measure, from the continuity of measures from below, we have:

$$\lim \int_{E_n} \varphi = \int \varphi$$

which establishes $\lim \int f_n \geq \alpha \int \varphi$. Since this result was independent of α, it remains true for $\alpha = 1$. Finally, as φ was any simple function such that $0 \leq \varphi \leq f$, we take the supremum of all possible φ to obtain $\lim \int f_n \geq \int f$. □

Theorem 6.2. *(Fatou's Lemma) Let $\{f_n\}$ be any sequence in L^+, we have:*

$$\int (lim\ inf\ f_n) \leq lim\ inf \int f_n.$$

Proof. For $k \geq 1$, $j \geq k$, we have $\inf_{n\geq k} f_n \leq f_j$. which implies $\int \inf_{n\geq k} f_n \leq$

$\int f_j$. Since this is true for all $j \geq k$, we must also have:

$$\int \text{inf}_{n\geq k} f_n \leq \text{inf}_{j\geq k} \int f_j$$

Note that on the left the infimum is over a sequence of functions, and on the right the infimum is over a sequence of real numbers. Now, we take $k \to \infty$ and apply the Monotone Convergence Theorem:

$$\int (\text{lim inf} f_n) = \lim_{k\to\infty} \int (\text{inf}_{n\geq k} f_n) \leq \text{lim inf} \int f_n.$$

□

Theorem 6.3. *(Dominated Convergence)*

Let $\{f_n\}$ *be any sequence in* L^1 *such that* $f_n \to f$ *a.e., and that there exists a nonnegative* $g \in L^1$ *such that* $|f_n| \leq g$ *a.e. for all* n*. Then* $f \in L^1$ *and* $\int f = \lim \int f_n$.

Proof. f is measurable because it is the limit of a sequence of measurable functions given a complete measure. Also, $|f| \leq g \in L^1$, which ensures that $\int |f| \leq \infty$. Therefore, we have $f \in L^1$. By treating the real and imaginary parts separately, it suffices to assume that f_n and f are real-valued. We have $g + f_n \geq 0$ and $g - f_n \geq 0$ a.e. By Fatou's lemma:

$$\int g + \int f \leq \text{lim inf} \int (g + f_n) = \int g + \text{lim inf} \int f_n$$

$$\int g - \int f \leq \text{lim inf} \int (g - f_n) = \int g - \text{lim sup} \int f_n$$

It follows that $\int f = \lim \int f_n$. □

7 Appendix B: Gallery of Zeros

In this section, we plot Z_n, the collection of zeros for all the possible Φ_n (See Chapter 5 for details). The plots will illustrate the symmetries proven in Theorems 5.17.

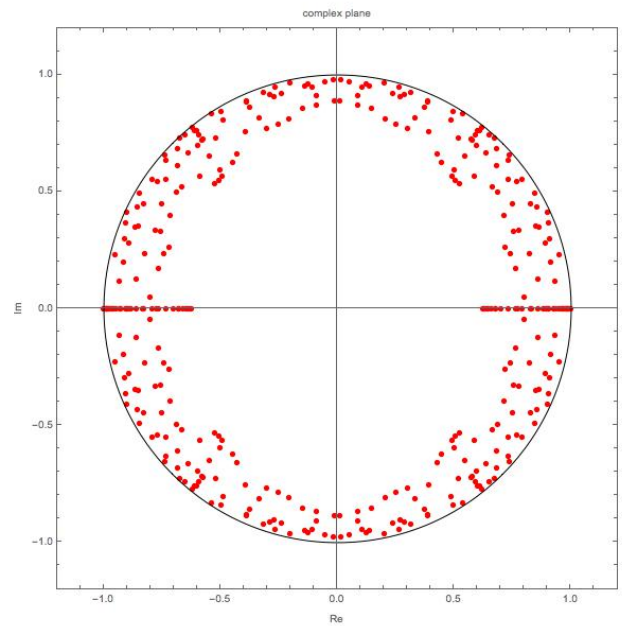

Figure 3: The collection of zeros Z_5 for $a = 1/2$, $b = -1/2$.

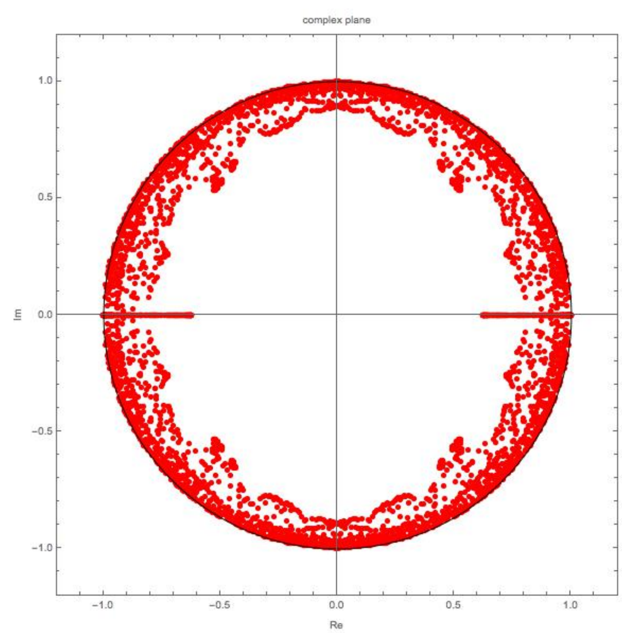

Figure 4: The collection of zeros Z_8 for $a = 1/2$, $b = -1/2$.

Since $a = -b$ is purely real, we see the symmetry about the origin. This is also true for purely imaginary $a = -b$, as below.

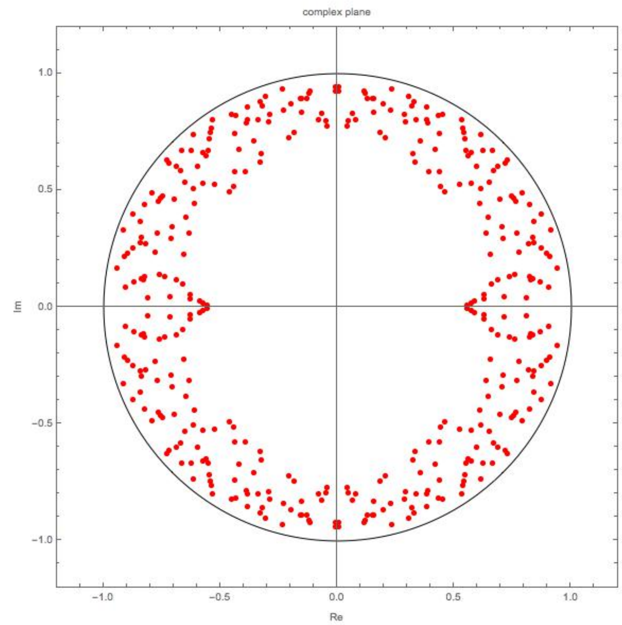

Figure 5: The collection of zeros Z_5 for $a = i/3$, $b = -i/3$.

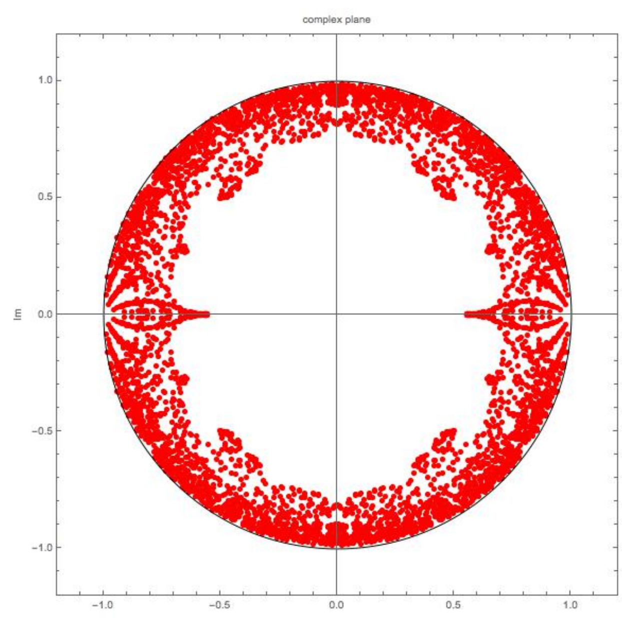

Figure 6: The collection of zeros Z_8 for $a = i/3$, $b = -i/3$.

But if a and b are not purely real or imaginary, then there is less symmetry.

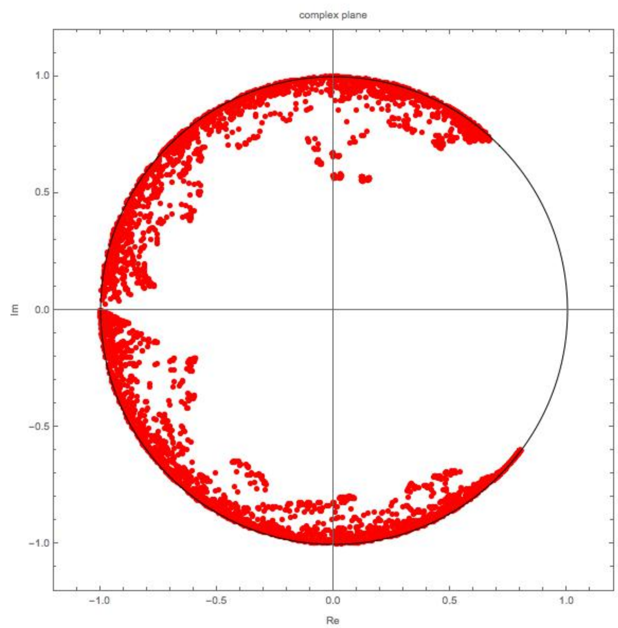

Figure 7: The collection of zeros Z_8 for $a = i/2 + 1/2$, $b = i/3 - 1/3$.

Similarly for $a = 0$ and $b = 1/2$. Note that the symmetry in across the real axis is simply because the Verblunsky coefficients are real. This gives monic OP's with only real coefficients, and the zeros consequently come in complex conjugate pairs.

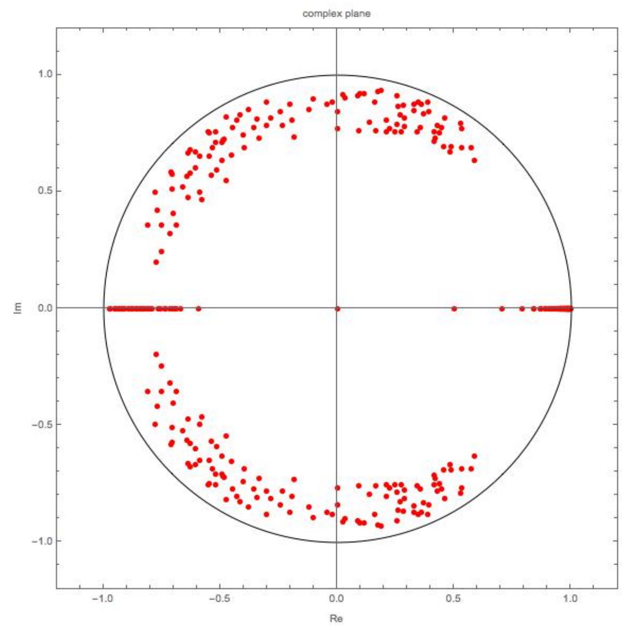

Figure 8: The collection of zeros Z_5 for $a = 0$, $b = 1/2$.

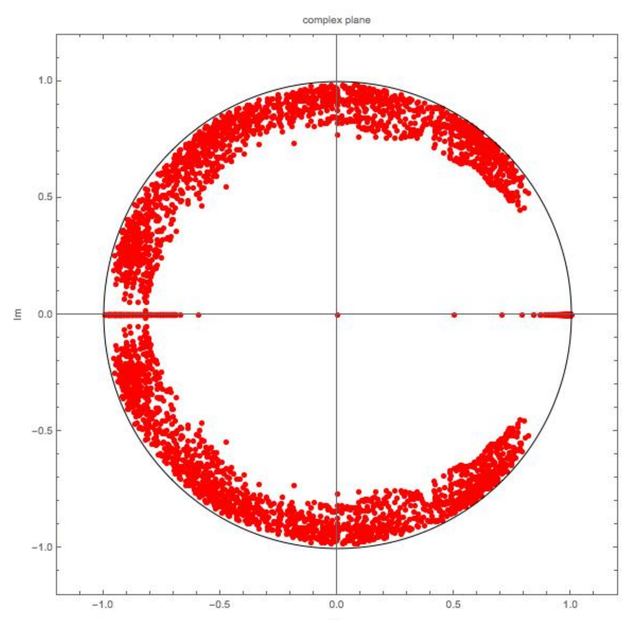

Figure 9: The collection of zeros Z_8 for $a = 0$, $b = 1/2$.

Finally, to illustrate that small perturbations in α_j leads to small perturbations in the zeros of the monic OP, consider the following figure. Here $a = 3/10$ and $b = 4/10$. We can think of any possible realization of the list of Verblunsky coefficients as a perturbation from the list with constant Verblunsky coefficient $\alpha = 3.5/10$. Each of the zeros are perturbed slightly in all possible realizations of Φ_5.

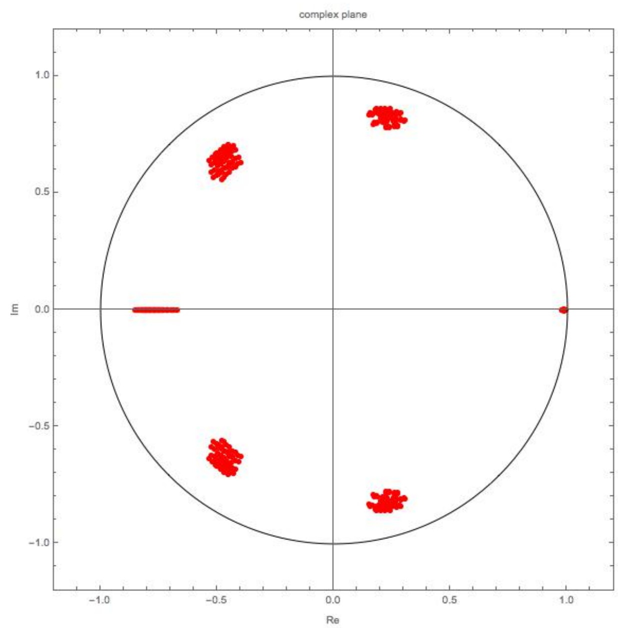

Figure 10: The collection of zeros Z_5 for $a = 3/10$, $b = 4/10$.

8 Appendix C: Mathematica Code

This section includes `Mathematica` implementation for computing various mathematical objects related to OPUC. For examples and simulations, see the two `Mathematica` notebooks. The numerics have been extremely useful in finding concrete examples and proving more general theorems.

```
(*Input: A list of (n+1) Verblunsky Coefficients. \[Alpha]0, \
\[Alpha]1, ..., \[Alpha]n*)
(*Output: \[CapitalPhi]_(n+1)*)

Recurrence[p_, deg_, c_] :=
```

```
 Expand[z*p - (c*z^deg*Conjugate[(p /. z -> (1/Conjugate[z]))])]
MonicOP[CV_List] := Module[{phi = 1, k = 0},
  While[k < Length[CV],
   phi = Recurrence[phi, k, Conjugate[CV[[k + 1]]]]; k++;];
  Return[phi];]

(*Input: Polynomial*)
(*Output: Reverse Polynomial, on the unit \
circle*)
ReversePoly[p_] :=
 Module[{coefs = CoefficientList[p, z], reverseC, reverseP},
  reverseC = Reverse[coefs] // Conjugate;
  reverseP = z^Range[0, (Length[reverseC] - 1)].reverseC;
  Return[reverseP];]

(*Input:Verblunsky Coefficients, \[Alpha]0, \[Alpha]1, ..., \[Alpha]n*)
(*Output: \[CapitalPhi]_n^*, the reverse OP*)
RMonicOP[CV_List] := ReversePoly[MonicOP[CV]]

(*Input:Verblunsky Coefficients, \[Alpha]0, ..., \[Alpha]n*)
\
(*Output: ||\[CapitalPhi]_{n+1}||^2 = \
||Reverse\[CapitalPhi]_{n+1}||^2*)

lengthPhi[CV_List] := Times @@ (1 - CV*Conjugate[CV])
(*Input:Verblunsky Coefficients, \[Alpha]0, ..., \[Alpha]n*)
\
(*Output: \[Phi]_{n+1}*)

OrthoOP[CV_List] := (MonicOP[CV])/Sqrt[lengthPhi[CV]] // FullSimplify
(*Input:Verblunsky Coefficients, \[Alpha]0, ..., \[Alpha]n*)
\
(*Output: Reverse\[Phi]_{n+1}*)
```

```
ROrthoOP[CV_List] := (RMonicOP[CV])/Sqrt[lengthPhi[CV]] //
  FullSimplify

(*Input:Subscript[\[Alpha], n],c1,..,cn*)
(*Output:Subscript[c, n+1]*)
\
(*This is inverting eq 1.5.80 to solve for c_{n+1}*)

helper[alphan_, moments_List] :=
  Module[{n = Length[moments], ans, numerator, denominator},
   numerator =
    ToeplitzMatrix[
     Join[{moments[[1]]}, {1}, Conjugate[moments[[1 ;; (n - 1)]]]],
     Join[moments, {x}]];
   denominator =
    ToeplitzMatrix[Join[{1}, Conjugate[moments]],
     Join[{1}, moments]];
   ans = x /.
     Solve[alphan == (-1)^n*Det[numerator]/Det[denominator], x][[1]];
   Return[ans]];

(*Now..ready to write the function which gives us the moments from \
Verb coeffs*)
(*Input:\[Alpha]0,...,\[Alpha]n*)
(*Output: List of c1, \
..., c(n+1)*)
(*Note, to handle corner case, we already know \
Subscript[\[Alpha], 0]=Subscript[c, 1]*)

VerbToMoments[alpha_List] :=
  Module[{moments = {alpha[[1]]}, n = Length[alpha]},
   Do[moments = Join[moments, {helper[alpha[[i]], moments]}],
    {i, 2, n}];
   Return[moments]];
```

```
(*Input:Subscript[c, 1],Subscript[c, 2], ..., Subscript[c, n], given \
we already know that Subscript[c, 0]=1,Subscript[c, \
-n]=Conjugate[Subscript[c, n]]*)
(*Output:(T^(n+1)) -- the (n+1) by \
(n+1) Toeplitz matrix, which has elements from Subscript[c, 0 ]to \
Subscript[c, n],Subscript[c, -n]*)

Toep[moments_List] :=
 ToeplitzMatrix[Join[{1}, Conjugate[moments]], Join[{1}, moments]]

(*Input:Subscript[c, 1],Subscript[c, 2], ..., Subscript[c, n]*)
\
(*This function implements Heine's Formula, see equation(1.5.77)*)
\
(*Output:Subscript[\[CapitalPhi], n]*)

MomentsToOP[moments_List] := Module[{a, b, c, n},
   n = Length[moments];
   a = (1/Det[Toep[Drop[moments, -1]]]);
   b = Transpose[Toep[moments]];(*Note the transpose is important*)

    b[[n + 1, All]] = Table[z^i, {i, 0, n}];
   c = Det[b];
   a*c
   ];

(*Input:c1, c2, ..., Subscript[c, n]*)
(*Output: Subscript[(\
\[CapitalPhi]^*), n], \[Phi]n,\[Phi]^*n*)

MomentsTophi[moments_List] :=
 Module[{P = MomentsToOP[moments], length, Dn, Dnm1},
  Dn = Det[Toep[moments]];
```

```
  Dnm1 = Det[Toep[Drop[moments, -1]]];
  length = Sqrt[Dn/Dnm1];
  Return[P/length];]
MomentsToROP[moments_List] := ReversePoly[MomentsToOP[moments]]
MomentsToRphi[moments_List] := ReversePoly[MomentsTophi[moments]]

(*Input:c1, c2, ..., Subscript[c, n]*)
(*Output:List of \[Alpha]0,\
\[Alpha]1,...,\[Alpha](n-1)*)

MomentsToVerb[moments_List] := Module[{result = {}},
   (*For every n, compute alpha_n = -Conjugate[Phi_{n+1}(0)]*)

   Do[result = Join[result, {-Conjugate[
         Coefficient[MomentsToOP[moments[[Range[i]]]], z, 0]]}],
    {i, Length[moments]}];
   Return[result]];

(*Input:c1,c2,...,Subscript[c, n]*)
(*Output: F and f, approximated \
from the list of moments*)

Carath\'edory[moments_List, z_] :=
 1 + 2*Sum[moments[[i]]*z^i, {i, 1, Length[moments]}]
Schur[moments_List,
  z_] := (1/
    z)*(Carath\'edory[moments, z] - 1)/(Carath\'edory[moments, z] + 1)

(*Input:c1, c2, ..., Subscript[c, n]*)
(*Output:An FINITE TAYLOR \
SERIES APPROXIMATION of w(\[Theta]) via Re(F(e^i\[Theta]))*)

density[moments_List, \[Theta]_] :=
 Re[1 + 2*Sum[
```

```
    moments[[i]]*Exp[I*i*\[Theta]], {i, 1, Length[moments]}]]

(*Input:w(\[Theta]), n*)
(*Output:c1, ..., Subscript[c, n], \
Integrated Analytically*)
(*NOTE: Follow the syntax in the example \
below, things can get weird because we're passing around functions as \
arguments*)
(*NOTE: Make sure there no singular part to the \
measure..or it doesn't work*)

DensityToMoments[w_, n_] := Module[{result = {}},
   Do[result =
     Join[result, {Integrate[
        Exp[-I*i*\[Theta]]*w[\[Theta]]/(2 \[Pi]), {\[Theta], 0,
         2 \[Pi]}]}],
    {i, n}];
   Return[result]];
(*SAME AS ABOVE, EXCEPT USE NINTEGRATE*)

NDensityToMoments[w_, n_] := Module[{result = {}},
   Do[result =
     Join[result, {NIntegrate[
        Exp[-I*i*\[Theta]]*w[\[Theta]]/(2 \[Pi]), {\[Theta], 0,
         2 \[Pi]}]}],
    {i, n}];
   Return[result]];

(*Input: Subscript[\[CapitalPhi], n]*)
(*Output: Plot of the zeros*)
\
(*NOTE: If we want the zeros from the Subscript[\[Alpha], n]'s or \
Subscript[c, n]'s, then just get the MonicOP from above and Plot*)
```

```
PlotZeros[P_, DotSize_] :=
 Module[{phi, solutions, solutionslist, pairslist, points},
  phi = P;
  solutions = Solve[phi == 0, z];
  solutionslist = z /. solutions;
  pairslist =
   Transpose[Join[{Re[solutionslist]}, {Im[solutionslist]}]];
  points =
   ListPlot[pairslist, PlotRange -> {{-1, 1}, {-1, 1}},
    AspectRatio -> 1,
    PlotStyle -> PointSize[DotSize]];
  Return[points]]

(*Input: Subscript[\[CapitalPhi], n]*)
(*Output: List of zeros as \
{Re,Im} pairs*)

ListZeros[P_] := Module[{phi, solutions, solutionslist, pairslist},
  phi = P;
  solutions = Solve[phi == 0, z];
  solutionslist = z /. solutions;
  pairslist =
   Transpose[Join[{Re[solutionslist]}, {Im[solutionslist]}]];
  Return[solutionslist // N]]
```