Clock Theorems for Eigenvalues of Finite CMV Matrices

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Abstract

We study probability measures on the unit circle and the distribution of the eigenvalues of the corresponding CMV matrices. We describe various classes of CMV matrices whose truncations exhibit a regular "clock" asymptotic eigenvalue distribution.

Contents

1 The Basics

1.1 Orthogonal Polynomials

Orthogonal polynomials for a given measure μ defined on $D \subset \mathbb{C}$ are simply polynomials that are pairwise orthogonal in $L^2(D, d\mu)$. We will be interested in the special case when $D = \partial\mathbb{D}$, the boundary of the unit disk, and μ is a nontrivial probability measure, i.e. $\mu(\partial\mathbb{D}) = 1$ and the support of μ is not a finite set. Critical to this theory is the fact that multiplication by z is unitary: We have

$$\langle zf, zg\rangle = \int_{\partial\mathbb{D}} \overline{(zf(z))} zg(z)\, d\mu = \int_{\partial\mathbb{D}} |z|^2 \overline{f(z)} g(z)\, d\mu = \langle f, g\rangle$$

since $|z| = 1$ on $\partial\mathbb{D}$; that it is onto is similarly easy to show.

We'll now use Gram-Schmidt to construct an orthonormal basis for the span of $\{1, z, z^2, \dots, z^n\}$. To make this precise we'll introduce the following notation:

Definition 1.1. *Let $e_1, \dots, e_n$ be linearly independent vectors with span M and choose $v \notin M$. Then there exist unique vectors $v_{||} \in M$ and $v_\perp \in M^\perp$ such that $v = v_{||} + v_\perp$. Define*

$$[v; e_1, \dots, e_n] = \frac{v_\perp}{\|v_\perp\|}, \qquad [\![v; e_1, \dots, e_n]\!] = v_\perp. \tag{1.1}$$

Definition 1.2. *Define the* orthogonal polynomials $\{\varphi_n\}_{n\geq 0}$ *and* monic orthogonal polynomails $\{\Phi_n\}_{n\geq 0}$ *by $\varphi_0 = \Phi_0 = 1$ and*

$$\Phi_n = [\![z^n; 1, z, \dots, z^{n-1}]\!], \qquad \varphi_n = [z^n; 1, z, \dots, z^{n-1}] \tag{1.2}$$

for $n > 0$.

Since $\{\varphi_1, \dots, \varphi_n\}$ is an orthonormal basis for the span of $\{1, z, \dots, z^n\}$ we have

$$\Phi_k(z) = z^k - \sum_{i=0}^{k-1} \langle \varphi_i, z^k\rangle \varphi_i. \tag{1.3}$$

Since the inner products in (1.3) will reduce to terms of the form

$$\langle z^i, z^j\rangle = \int \overline{z}^i z^j\, d\mu = \int z^{j-i}\, d\mu$$

it is useful to make the following definition:

Definition 1.3. *For a measure μ define coefficients c_n, called the* moments *of μ, by*

$$c_n = \int z^{-n} \, d\mu. \tag{1.4}$$

Definition 1.4. *Let the map $R_n \colon L^2(\partial\mathbb{D}, d\mu) \to L^2(\partial\mathbb{D}, d\mu)$ be defined by*

$$R_n(f)(z) = z^n \overline{f(z)}. \tag{1.5}$$

If p is a polynomial of degree d, we'll write p^* for $R_d(p)$.

If $p(z) = \sum_{i=0}^{n} a_i z^i$ is an nth degree polynomial, we can write an alternate definition of p^* in terms of its coefficients:

$$p^*(z) = z^n \sum_{i=0}^{n} \overline{a}_i \overline{z}^i = \sum_{i=0}^{n} \overline{a}_i (z^n \overline{z}^i) = \sum_{i=0}^{n} \overline{a}_i z^{n-i} = \sum_{i=0}^{n} \overline{a}_{n-i} z^i. \tag{1.6}$$

Definition 1.5. *An onto map $U \colon V \to V$ is called antiunitary if for all $f, g \in V$ and $\lambda \in \mathbb{C}$ we have*

1. *$U(f+g) = Uf + Ug$*
2. *$U(\lambda f) = \overline{\lambda} U f$*
3. *$\langle Uf, Ug \rangle = \langle g, f \rangle$.*

Lemma 1.6. *The map R_n is antiunitary.*

Proof. R_n is onto since for all $f \in L^2(\partial\mathbb{D}, d\mu)$ we have $g = z^{-n}\overline{f} \in L^2(\partial\mathbb{D}, d\mu)$ and $R_n g = f$. (1) and (2) are clear from the definition. To see (3) we note that

$$\begin{aligned} \langle R_n f, R_n g \rangle &= \int \overline{z^n \overline{f(z)}} \cdot z^n \overline{g(z)} \, d\mu = \int \overline{z}^n z^n \overline{g(z)} f(z) \, d\mu \\ &= \int \overline{g(z)} f(z) \, d\mu = \langle g, f \rangle. \end{aligned} \tag{1.7}$$

□

Using this property of R_n we can prove the following theorem:

Theorem 1.7. *(Szegő Recursion) For any nontrivial probability measure μ on $\partial\mathbb{D}$ there is a unique sequence of numbers $\{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{D}$, called Verblunsky coefficients, such that for all n*

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z). \tag{1.8}$$

Proof. Clearly $z\Phi_n - \overline{\alpha}_n \Phi_n^*$ is a monic polynomial of degree $n+1$, so all that remains to show is that it is orthogonal to $\{1, z, \ldots, z^n\}$. Let $1 \le k \le n$. Then

$$\begin{aligned}\langle z^k, z\Phi_n - \overline{\alpha}_n \Phi_n^* \rangle &= \langle z^k, z\Phi_n \rangle - \overline{\alpha}_n \langle z^k, \Phi_n^* \rangle \\ &= \langle z^{k-1}, \Phi_n \rangle - \overline{\alpha}_n \langle R_n(\Phi_n^*), R_n(z^k) \rangle \\ &= \langle z^{k-1}, \Phi_n \rangle - \overline{\alpha}_n \langle \Phi_n, z^{n-k} \rangle \\ &= 0\end{aligned}$$

since $\Phi_n \perp z^j$ for $0 \le j < n$. Requiring orthogonality for $k = 0$ gives

$$\begin{aligned}0 &= \langle 1, z\Phi_n - \overline{\alpha}_n \Phi_n^* \rangle \\ \langle 1, z\Phi_n \rangle &= \langle 1, \overline{\alpha}_n \Phi_n^* \rangle \\ \int z\Phi_n(z)\, d\mu &= \overline{\alpha}_n \int \Phi_n^*(z)\, d\mu \\ \overline{\alpha}_n &= \frac{\int z\Phi_n(z)\, d\mu}{\int \Phi_n^*(z)\, d\mu}. \end{aligned} \tag{1.9}$$

Using (1.9) as the definition for α_n we get $(z\Phi_n - \overline{\alpha}_n \Phi_n^*) \perp z^k$ for $0 < k < n$ and hence $z\Phi_n - \overline{\alpha}_n \Phi_n^* = \Phi_{n+1}$. Rearranging (1.8) we get $z\Phi_n = \Phi_{n+1} + \overline{\alpha}_n \Phi_n$. This, together with $\Phi_{n+1} \perp \Phi_n^*$ (since the degree of Φ_n^* is $\le n$) and the Pythagorean theorem gives

$$\|z\Phi_n\|^2 = \|\Phi_{n+1}\|^2 + |\alpha_n|^2 \|\Phi_n^*\|^2 \tag{1.10}$$

Since multiplication by z is unitary and R_n is antiunitary we have $\|z\Phi_n\| = \|\Phi_n\| = \|\Phi_n^*\|$. Therefore (1.10) becomes

$$\|\Phi_{n+1}\|^2 = (1 - |\alpha_n|^2) \|\Phi_n\|^2 \tag{1.11}$$

which shows that $|\alpha_n| < 1$ (since we must have $\|\Phi_{n+1}\| > 0$) and hence $\alpha_n \in \mathbb{D}$. □

There are many interesting identities involving Verblunsky coefficients and orthogonal polynomials. Firstly, we can easily recover α_n from Φ_{n+1}: setting $z = 0$ in (1.8) gives

$$\begin{aligned}\Phi_{n+1}(0) &= z\Phi_n(0) - \overline{\alpha}_n \Phi_n^*(0) \\ &= 0 - \overline{\alpha}_n\end{aligned}$$

and hence

$$\alpha_n = -\overline{\Phi_{n+1}(0)}. \tag{1.12}$$

We can also express $\|\Phi_n\|$ in terms of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. Define new coefficients $\rho_k = \sqrt{1 - |\alpha_n|^2}$. Then induction on (1.11) gives

$$\|\Phi_n\| = \rho_0 \rho_1 \cdots \rho_{n-1}. \tag{1.13}$$

Combining this and (1.8) we get the recurrence relation for the φ's:

$$\rho_n \varphi_{n+1}(z) = z\varphi_n(z) - \overline{\alpha}_n \varphi_n^*(z).$$

Finally, applying R_{n+1} to the above equation and doing basic algebraic maniuplation (see page 6 of [6]) yields four relations:

$$\rho_n \varphi_{n+1}(z) = z\varphi_n(z) - \overline{\alpha}_n \varphi_n^*(z) \tag{1.14}$$
$$\rho_n \varphi_{n+1}^*(z) = \varphi_n^*(z) - \alpha_n z \varphi_n(z) \tag{1.15}$$
$$\rho_n z \varphi_n(z) = \varphi_{n+1}(z) + \overline{\alpha}_n \varphi_{n+1}^*(z) \tag{1.16}$$
$$\rho_n \varphi_n^*(z) = \varphi_{n+1}^*(z) + \alpha_n \varphi_{n+1}(z). \tag{1.17}$$

Example 1.8. Let $d\lambda$ be Lebesgue measure on $\partial\mathbb{D}$, that is $d\lambda = \frac{d\theta}{2\pi}$. We have

$$c_n = \int_{\partial\mathbb{D}} z^{-n} \, d\lambda = \frac{1}{2\pi} \int_{\partial\mathbb{D}} e^{-in\theta} d\theta = \delta_{n0}.$$

This gives $\langle z^i, z^j \rangle = 0$ for $i \neq j$ and hence

$$\Phi_n(z) = z^n - \sum_{i=0}^{n-1} \langle \varphi_i, z^n \rangle \varphi_i = z^n$$

since the degree of φ_i is less than n. $\|z^n\| = 1$ implies $\varphi_n = \Phi_n = z^n$. By (1.12) we have that for all n

$$\alpha_n = -\overline{\Phi_{n+1}(0)} = 0$$

and hence $\rho_n = 1$.

1.2 The CMV Basis

Unfortunately, $\{\varphi_n\}$ is not necessarily a basis of $L^2(\partial\mathbb{D}, d\mu)$ (in fact, it is a basis if and only if $\sum_{n=0}^{\infty} |\alpha_n| = \infty$, see [4]). Since the Laurent polynomials are always dense in $C(\partial\mathbb{D})$ and hence in $L^2(\partial\mathbb{D}, d\mu)$, we'll instead apply Gram-Schmidt to the polynomials $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$ and $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$, to get the *CMV basis* $\{\chi_n\}_{n=0}^{\infty}$ and *alternate CMV basis* $\{x_n\}_{n=0}^{\infty}$, respectively:

$$\chi_{2k-1} = [z^k; 1, z, z^{-1}, \ldots, z^{k-1}, z^{-k+1}] \quad \chi_{2k} = [z^{-k}; 1, z, z^{-1}, \ldots, z^{-k+1}, z^k]$$
$$x_{2k-1} = [z^{-k}; 1, z^{-1}, z, \ldots, z^{-k+1}, z^{k-1}] \quad x_{2k} = [z^k; 1, z^{-1}, z, \ldots, z^{k-1}, z^{-k}].$$

We eventually want to write the matrix for multiplication by z in the CMV basis, called the *CMV matrix*. Our immediate goal, however, will be to express the CMV and alternate CMV bases in terms of one another.

Lemma 1.9. *Let V be the span of normal vectors $e_1, \ldots, e_n$. If $U\colon V \to V$ is either unitary or antiunitary, then for all $v \in V$ we have*

$$U([\![v; e_1, \ldots, e_n]\!]) = [\![Uv; Ue_1, \ldots, Ue_n]\!], \tag{1.18}$$
$$U([v; e_1, \ldots, e_n]) = [Uv; Ue_1, \ldots, Ue_n]. \tag{1.19}$$

Proof. First we'll use Gram-Schmidt to find an orthonormal basis $\{w_1, \ldots, w_n\}$ for span$\{e_1, \ldots, e_n\}$. Routine linear algebra arguments show that that $\{Uw_1, \ldots, Uw_n\}$ is an orthonormal basis for span$\{Ue_1, \ldots, Ue_n\}$. Thus

$$[\![v; e_1, \ldots, e_n]\!] = [\![v; w_1, \ldots, w_n]\!] = v - \sum_{i=1}^{n} \langle w_i, v\rangle w_i$$

and

$$[\![Uv; Ue_1, \ldots, Ue_n]\!] = [\![Uv; Uw_1, \ldots, Uw_n]\!] = U - \sum_{i=1}^{n} \langle Uw_i, Uv\rangle Uw_i.$$

If U is unitary we get

$$\begin{aligned}
[\![Uv; Ue_1, \ldots, Ue_{m+1}]\!] &= U\left(v - \sum_{j=1}^{m+1} \langle Uw_j, Uv\rangle w_j\right) \\
&= U\left(v - \sum_{j=1}^{m+1} \langle w_j, v\rangle w_j\right) \\
&= U([\![v; e_1, \ldots, e_{m+1}]\!])
\end{aligned}$$

while if U is antiunitary we get

$$\begin{aligned}
[\![Uv; Ue_1, \dots, Ue_{m+1}]\!] &= U\left(v - \sum_{j=1}^{m+1} \overline{\langle Uw_j, Uv\rangle} w_j\right) \\
&= U\left(v - \sum_{j=1}^{m+1} \langle w_j, v\rangle w_j\right) \\
&= U([\![v; e_1, \dots, e_{m+1}]\!]).
\end{aligned}$$

In either case U preserves lengths so we also get

$$[Uv; Ue_1, \dots, Ue_{m+1}] = U([v; e_1, \dots, e_{m+1}]).$$

□

Since $\|z^n\| = 1$ for all n we can apply Lemma 1.9 to the definitions of our various polynomials. Applying R_n to the definition of φ_n gives

$$\varphi_n^* = [R_n z^n; R_n(1), R_n(z), \dots, R_n(z^{n-1})] \tag{1.20}$$
$$= [1; z, z^2, \dots, z^n]. \tag{1.21}$$

We can also pull out a z^{-k} from the defition of x_{2k} to get

$$\begin{aligned}
x_{2k} &= [z^k; 1, z^{-1}, z, \dots, z^{k-1}, z^{-k}] \\
&= z^{-k}[z^{2k}; z^k, z^{k-1}, z^{k+1}, z^{k-2} z^{k+2}, \dots, z^2, z^{2k-2}, z, z^{2k-1}] \\
&= z^{-k}[z^{2k}; 1, z, z^2, \dots, z^{2k-2}, z^{2k-1}] \\
&= z^{-k} \varphi_{2k}.
\end{aligned}$$

Similar calculations yield

$$\chi_{2k-1}(z) = z^{-k+1}\varphi_{2k-1}(z) \qquad \chi_{2k}(z) = z^{-k}\varphi_{2k}^*(z) \tag{1.22}$$
$$x_{2k-1}(z) = z^{-k}\varphi_{2k-1}^*(z) \qquad x_{2k}(z) = z^{-k}\varphi_{2k}(z). \tag{1.23}$$

Now (1.15) for $n = 2k-1$ gives

$$\varphi_{2k}^* = \rho_{2k-1}\varphi_{2k-1}^* - \alpha_{2k-1}\varphi_{2k}.$$

Multiplying through by z^{-k} and applying (1.22) and (1.23), we obtain

$$\chi_{2k} = \rho_{2k-1} x_{2k-1} - \alpha_{2k-1} x_{2k}.$$

Similarly combining (1.22) and (1.23) with different combinations of (1.14)-(1.17), we get four new relations:

$$\chi_{2k-1} = \overline{\alpha}_{2k-1}x_{2k-1} + \rho_{2k-1}x_{2k} \qquad \chi_{2k} = \rho_{2k-1}x_{2k-1} - \alpha_{2k-1}x_{2k}$$
$$zx_{2k} = \overline{\alpha}_{2k}\chi_{2k} + \rho_{2k}\chi_{2k+1} \qquad zx_{2k+1} = \rho_{2k}\chi_{2k} - \alpha_{2k}\chi_{2k+1}.$$

These can be written more compactly as

$$\begin{pmatrix}\chi_{2k-1}\\ \chi_{2k}\end{pmatrix} = \Theta_{2k-1}\begin{pmatrix}x_{2k-1}\\ x_{2k}\end{pmatrix}, \qquad \begin{pmatrix}zx_{2k}\\ zx_{2k+1}\end{pmatrix} = \Theta_{2k}\begin{pmatrix}\chi_{2k}\\ \chi_{2k+1}\end{pmatrix} \tag{1.24}$$

where

$$\Theta_n = \begin{pmatrix}\overline{\alpha}_n & \rho_n\\ \rho_n & -\alpha_n\end{pmatrix}. \tag{1.25}$$

Example 1.10. If $d\mu = d\lambda$ is the Lebesgue measure we have

$$\chi_{2k-1}(z) = x_{2k}(z) = z^k \qquad \chi_{2k}(z) = x_{2k-1}(z) = z^{-k}.$$

and

$$\Theta_n \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

1.3 The CMV Matrix

Now we're ready to write the *CMV matrix* $\mathcal{C}$, the matrix for T_z in the CMV basis. For a linear transformation $T\colon V \to V$ and bases B_1 and B_2 of V let $[T]_{B_1}^{B_2}$ denote the matrix representation of A with domain expressed in B_1 and range expressed in B_2. For example in this notation the change of basis matrix going from B_1 to B_2 would be $[I]_{B_1}^{B_2}$. For linear transformations $T, S\colon V \to V$ and bases B_1, B_2, B_3 of V we have

$$[TS]_{B_1}^{B_3} = [T]_{B_2}^{B_3}[S]_{B_1}^{B_2}. \tag{1.26}$$

Let $B_\chi = \{\chi_n\}_{n\geq 0}$ and $B_x = \{x_n\}_{n\geq 0}$. Defining matrices $\mathcal{L}$ and $\mathcal{M}$ by

$$\mathcal{L} = [T_z]_{B_x}^{B_\chi} \qquad \mathcal{M} = [I]_{B_\chi}^{B_x}. \tag{1.27}$$

we see that (1.26) gives

$$\mathcal{C} = [T_z]_{B_\chi}^{B_\chi} = [T_z I]_{B_\chi}^{B_\chi} = [T_z]_{B_x}^{B_\chi}[I]_{B_\chi}^{B_x} = \mathcal{L}\mathcal{M}. \tag{1.28}$$

Since

$$\chi_j = \sum_{i=0}^{\infty} \langle x_i, \chi_j \rangle x_i \qquad\qquad zx_j = \sum_{i=0}^{\infty} \langle \chi_i, zx_j \rangle \chi_i$$

we have

$$\mathcal{M}_{ij} = \langle x_i, \chi_j \rangle \qquad\qquad \mathcal{L}_{ij} = \langle \chi_i, zx_j \rangle. \tag{1.29}$$

Using (1.24) and the fact that $x_0 = \chi_0 = 1$, this gives

$$\mathcal{L} = \Theta_0 \oplus \Theta_2 \oplus \Theta_4 \oplus \Theta_6 \oplus \cdots \tag{1.30}$$

$$\mathcal{M} = I_{1\times 1} \oplus \Theta_1 \oplus \Theta_3 \oplus \Theta_5 \oplus \cdots . \tag{1.31}$$

Doing the matrix multiplication gives

$$\mathcal{C} = \begin{pmatrix} \overline{\alpha}_0 & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \cdots \\ \rho_0 & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \cdots \\ 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 & \rho_3\rho_2 & \cdots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \cdots \\ 0 & 0 & 0 & \overline{\alpha}_4\rho_3 & -\overline{\alpha}_4\alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.32}$$

$$= \begin{pmatrix} \widetilde{B}_{-1} & B_0^\top & & & & \\ & B_1 & B_2^\top & & & \\ & & B_3 & B_4^\top & & \\ & & & B_5 & B_6^\top & \\ & & & & \ddots & \ddots \end{pmatrix} \tag{1.33}$$

where

$$B_k = \begin{pmatrix} \overline{\alpha}_{k+1}\rho_k & -\overline{\alpha}_{k+1}\alpha_k \\ \rho_{k+1}\rho_k & -\rho_{k+1}\alpha_k \end{pmatrix} \qquad\qquad \widetilde{B}_{-1} = \begin{pmatrix} \overline{\alpha}_0 \\ \rho_0 \end{pmatrix}. \tag{1.34}$$

In summary, we get

Theorem 1.11. *Let $\mathcal{C}$ be the matrix for multiplaction in the $\{\chi_n\}_{n\geq 0}$ basis. Then $\mathcal{C} = \mathcal{L}\mathcal{M}$ with $\mathcal{L}$ and $\mathcal{M}$ defined as in* (1.30) *and* (1.31)*. Furthermore, $\mathcal{C}$ has the form given in* (1.32) *and* (1.33)*.*

Now (1.33) tells us that $\mathcal{C}$ has five nonzero diagonals. It's interesting to note that we could have deduced this only knowing that $\mathcal{L}$ and $\mathcal{M}$ had two nonzero diagonals:

Lemma 1.12. *Let A be a matrix with $2m+1$ nonzero diagonals, and B a matrix with $2n+1$ diagonals. Then AB has $2(m+n)+1$ nonzero diagonals.*

Proof. Suppose A has $2m+1$ diagonals, and B $2n+1$ diagonals. Let $C = AB$. Then for the ijth term of C, $c_{ij} = 0$ if $|i-j| > m+n$. To see this, note that c_{ij} is the dot product of the ith row of A and the jth column of B. Now, $a_{ik} \neq 0$ if $|k-i| \leq m$, and $b_{hj} \neq 0$ if $|h-j| \leq n$. Then all the products $a_{ir}b_{rj}$ are zero except when $|r-i| \leq m$ and $|r-j| \leq n$. But if this holds, then $|i-j| \leq m+n$. □

1.4 Finite and Cutoff CMV Matrices

In this section we will discuss two finite-dimensional matrices that we can construct from a CMV matrix.

Definition 1.13. *For complex numbers $\beta_0, \ldots, \beta_n \in \overline{\mathbb{D}}$ let $\mathcal{C}(\beta_0, \ldots, \beta_n)$ be the upper $(n+1) \times (n+1)$ block of the matrix given by formally substituting $\alpha_k = \beta_k$ and $\rho_k = \sqrt{1-|\beta_k|^2}$ in* (1.32).

Example 1.14. For $\alpha_0, \ldots, \alpha_3 \in \overline{\mathbb{D}}$ and $\rho_k = \sqrt{1-|\alpha_k|^2}$ we get

$$\mathcal{C}(\alpha_0, \ldots, \alpha_3) = \begin{pmatrix} \overline{\alpha}_0 & \overline{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 \\ \rho_0 & -\overline{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 \\ 0 & \overline{\alpha}_2\rho_1 & -\overline{\alpha}_2\alpha_1 & \overline{\alpha}_3\rho_2 \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha}_3\alpha_2 \end{pmatrix}. \tag{1.35}$$

For a CMV matrix $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots)$ we define the *cutoff CMV matrix* $\mathcal{C}^{(n)}$ by

$$\mathcal{C}^{(n)} = \mathcal{C}(\alpha_0, \ldots, \alpha_{n-1}), \tag{1.36}$$

and for $\beta \in \partial\mathbb{D}$ we define the *finite CMV matrix* $\mathcal{C}_\beta^{(n)}$ by

$$\mathcal{C}_\beta^{(n)} = \mathcal{C}(\alpha_0, \ldots, \alpha_{n-2}, \beta). \tag{1.37}$$

2 Lebesgue and Inserted Mass Point

2.1 Finite Cutoffs of Lebesgue

Let us look at finite cutoff matrices of Λ, the CMV matrix associated with the Lebesgue measure.

Theorem 2.1. *Every eigenvalue of $\Lambda_\beta^{(n)}$ is an nth root of $\overline{\beta}$.*

Proof. By equation (1.18), we know that $\Phi_{n-1}(z) = z^{n-1}$ since $\alpha_k = 0$ for $0 \leq k \leq n-2$. Hence

$$\begin{aligned}\Phi_n(z) &= z \cdot \Phi_{n-1}(z) - \overline{\beta} \cdot \Phi^*_{n-1}(z) \\ &= z^n - \overline{\beta} \cdot 1 \\ &= z^n - \overline{\beta}\end{aligned}$$

But we know from [6] that the eigenvalues of a finite matrix are exactly the zeros of the paraorthogonal polynomial. Hence if λ is an eigenvalue of $\Lambda^{(n)}_{\beta}$, then $\lambda^n - \overline{\beta} = 0$. Therefore $\lambda^n = \overline{\beta}$. $\square$

2.2 Single Inserted Mass Point

Let us look at the case where our measure is

$$d\mu_{e^{i\theta},\gamma} = (1-\gamma)d\lambda + \gamma\delta_{e^{i\theta}}$$

where λ is the Lebesgue measure and $\delta_{e^{i\theta}}$ is the dirac measure centered at $e^{i\theta}$. This is indeed a probability measure since

$$\begin{aligned}\int_{\partial\mathbb{D}} d\mu_{e^{i\theta},\gamma} &= (1-\gamma)\int_{\partial\mathbb{D}} d\lambda + \gamma \int_{\partial\mathbb{D}} \delta_{e^{i\theta}} = (1-\gamma) + \gamma \\ &= 1\end{aligned}$$

as both dz and $\delta_{e^{i\theta}}$ are probability measures. This measure is often referred to as a "single inserted mass point" and we can find the Verblunsky coefficients α_n explicitly for this example.

Theorem 2.2. *If $d\mu_{\omega,\gamma} = (1-\gamma)d\lambda + \gamma\delta_\omega$ where $\omega \in \partial\mathbb{D}$ then*

$$\Phi_n(z) = z^n - \frac{\gamma}{1+n\gamma-\gamma}(\omega z^{n-1} + \omega^2 z^{n-2} + \cdots + \omega^n) \tag{2.1}$$

and

$$\alpha_n = \frac{\omega^{-(n+1)}\gamma}{1+n\gamma}. \tag{2.2}$$

Proof. We will begin by finding the moments associated with $d\mu_{\theta,\gamma}$:

$$
\begin{aligned}
c_n &= \int_{\partial\mathbb{D}} z^{-n} d\mu_{\theta,\gamma} \\
&= (1-\gamma)\int_{\partial\mathbb{D}} z^{-n} d\lambda + \gamma \int_{\partial\mathbb{D}} z^{-n}\delta_\omega \\
&= \frac{1-\gamma}{2\pi}\int_{[0,2\pi]} e^{-in\theta} d\theta + \gamma\omega^{-n} \\
&= \frac{1-\gamma}{-2i\pi n} e^{-in\theta}\Big|_0^{2\pi} + \gamma\omega^{-n} \\
&= \gamma\omega^{-n}
\end{aligned}
$$

if $n \neq 0$. Hence we have

$$
c_n = \begin{cases} 1 & \text{if } n = 0; \\ \gamma\omega^{-n} & \text{if } n \neq 0. \end{cases}
$$

We know from [4] that if $L_n = T^{(n+1)\top}$ where $T^{(n+1)}$ is the Toeplitz matrix, in other words

$$
L_n = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_2 & c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & c_{n-2} & \cdots & c_0 \end{pmatrix}
$$

then the vector $\mathbf{a}^{(n)} = \langle \delta_n, L_n^{-1}\delta_n\rangle^{-1} L_n^{-1}\delta_n$ where $\delta_n = (0,0,\ldots,1)^\top$ will be exactly the coefficients of Φ_n, the degree n monic orthogonal polynomial. In other words, if $\mathbf{a}^{(n)} = (a_0, a_1, \ldots, a_n)^\top$ then $\Phi_n(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Note that in our case,

$$
\begin{aligned}
L_n &= \begin{pmatrix} 1 & \gamma\omega & \gamma\omega^2 & \cdots & \gamma\omega^n \\ \gamma\omega^{-1} & 1 & \gamma\omega & \cdots & \gamma\omega^{n-1} \\ \gamma\omega^{-2} & \gamma\omega^{-1} & 1 & \cdots & \gamma\omega^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma\omega^{-n} & \gamma\omega^{-n+1} & \gamma\omega^{-n+2} & \cdots & 1 \end{pmatrix} \\
&= (1-\gamma)I + \gamma P
\end{aligned}
$$

where

$$P = \begin{pmatrix} 1 & \omega & \omega^2 & \cdots & \omega^n \\ \omega^{-1} & 1 & \omega & \cdots & \omega^{n-1} \\ \omega^{-2} & \omega^{-1} & 1 & \cdots & \omega^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{-n} & \omega^{-n+1} & \omega^{-n+2} & \cdots & 1 \end{pmatrix}.$$

Note that P is a rank 1 matrix since all columns are multiples of the first column. Hence the column space has dimension 1. By the rank-nullity theorem, this means that the nullspace of P has dimension n, since P is an $(n+1)\times(n+1)$ matrix. Hence there is an n-dimensional subspace $V \subset \mathbb{C}^{n+1}$ such that $P\mathbf{v} = \mathbf{0}\ \forall \mathbf{v} \in V$. Hence if $\mathbf{v} \in V$

$$L_n\mathbf{v} = (1-\gamma)\mathbf{v} + \gamma P\mathbf{v} = (1-\gamma)\mathbf{v}$$

Thus L_n has $(1-\gamma)$ as an eigenvalue with geometric multiplicity n. Now consider the vector $\mathbf{w} = (\omega^n, \omega^{n-1}, \cdots, \omega, 1)^\top$. A simple computation shows that $P\mathbf{w} = ((n+1)\omega^n, (n+1)\omega^{n-1}, \cdots, (n+1)\omega, (n+1))^\top = (n+1)\mathbf{w}$. Hence

$$L_n\mathbf{w} = (1-\gamma)\mathbf{w} + \gamma(n+1)\mathbf{w} = (1+n\gamma)\mathbf{w}$$

Thus $(1+n\gamma)$ is the final eigenvalue of L_n.

Since L_n is an $(n+1)\times(n+1)$ matrix with $n+1$ linearly independent eigenvectors, it is diagonalizable, and hence its minimal polynomial is simply a product of linear factors corresponding to distinct eigenvalues. In other words if $m(x)$ is the minimal polynomial of L_n, then

$$\begin{aligned} m(x) &= (x-(1-\gamma))(x-(1+n\gamma)) \\ &= x^2 - (2-\gamma+n\gamma)x + (1-\gamma)(1+n\gamma). \end{aligned}$$

Hence

$$L_n^2 - (2-\gamma+n\gamma)L_n = -(1-\gamma)(1+n\gamma)I$$

and therefore

$$\begin{aligned} L_n^{-1} &= -(1-\gamma)^{-1}(1+n\gamma)^{-1}[L_n - (2-\gamma+n\gamma)I] \\ &= -(1-\gamma)^{-1}(1+n\gamma)^{-1}[(1-\gamma)I + \gamma P - (2-\gamma+n\gamma)I] \\ &= -(1-\gamma)^{-1}(1+n\gamma)^{-1}[\gamma P - (1+n\gamma)I] \\ &= (1-\gamma)^{-1}I - \frac{\gamma}{(1-\gamma)(1+n\gamma)}P. \end{aligned}$$

With this formula for L_n^{-1} we can now calculate Φ_n. We know that $I\delta_n = \delta_n$ and $P\delta_n = (\omega^n, \omega^{n-1}, \cdots, \omega, 1)^\top = \mathbf{w}$, so

$$\begin{aligned}
L_n^{-1}\delta_n &= (1-\gamma)^{-1}\delta_n - \frac{\gamma}{(1-\gamma)(1+n\gamma)}\mathbf{w} \\
&= \begin{pmatrix} -\frac{\gamma\omega^n}{(1-\gamma)(1+n\gamma)} \\ -\frac{\gamma\omega^{n-1}}{(1-\gamma)(1+n\gamma)} \\ \vdots \\ -\frac{\gamma\omega}{(1-\gamma)(1+n\gamma)} \\ \frac{1}{1-\gamma} - \frac{\gamma}{(1-\gamma)(1+n\gamma)} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\gamma\omega^n}{(1-\gamma)(1+n\gamma)} \\ -\frac{\gamma\omega^{n-1}}{(1-\gamma)(1+n\gamma)} \\ \vdots \\ -\frac{\gamma\omega}{(1-\gamma)(1+n\gamma)} \\ \frac{1+n\gamma-\gamma}{(1-\gamma)(1+n\gamma)} \end{pmatrix}
\end{aligned}$$

Hence $\langle \delta_n, L_n^{-1}\delta_n \rangle = \frac{1+n\gamma-\gamma}{(1-\gamma)(1+n\gamma)}$. Now we can calculate $\mathbf{a}^{(n)}$:

$$\begin{aligned}
\mathbf{a}^{(n)} &= \langle \delta_n, L_n^{-1}\delta_n \rangle^{-1} L_n^{-1}\delta_n \\
&= \frac{(1-\gamma)(1+n\gamma)}{1+n\gamma-\gamma} \begin{pmatrix} -\frac{\gamma\omega^n}{(1-\gamma)(1+n\gamma)} \\ -\frac{\gamma\omega^{n-1}}{(1-\gamma)(1+n\gamma)} \\ \vdots \\ -\frac{\gamma\omega}{(1-\gamma)(1+n\gamma)} \\ \frac{1+n\gamma-\gamma}{(1-\gamma)(1+n\gamma)} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\gamma\omega^n}{1+n\gamma-\gamma} \\ -\frac{\gamma\omega^{n-1}}{1+n\gamma-\gamma} \\ \vdots \\ -\frac{\gamma\omega}{1+n\gamma-\gamma} \\ 1 \end{pmatrix}.
\end{aligned}$$

Hence we have

$$\Phi_n(z) = z^n - \frac{\gamma}{1+n\gamma-\gamma}(\omega z^{n-1} + \cdots + \omega^{n-1}z + \omega^n) \qquad (2.3)$$

as promised.

All that remains is to calculate α_n, but by (1.12) we know that

$$\begin{aligned}\alpha_{n-1} &= -\overline{\Phi_n(0)} \\ &= \frac{\overline{\omega^n}\gamma}{1+n\gamma-\gamma} \\ &= \frac{\omega^{-n}\gamma}{1+n\gamma-\gamma}\end{aligned}$$

since $\omega \in \partial\mathbb{D}$. Hence

$$\begin{aligned}\alpha_n &= \frac{\omega^{-(n+1)}\gamma}{1+(n+1)\gamma-\gamma} \\ &= \frac{\omega^{-(n+1)}\gamma}{1+n\gamma}\end{aligned}$$

as desired. □

Now we want to consider finite cutoff matrices of the CMV matrices associated with the single inserted mass point measures. As discussed above, the eigenvalues of $\Lambda_\beta^{(n)}$ are simply the nth roots of β. We would like to see how the single inserted mass point relates to this simpler case.

Theorem 2.3. *Let $\mathcal{C} = \mathcal{C}(d\mu_{\omega,\gamma})$. Then if $\omega^n = \overline{\beta}$, the eigenvalues of $\mathcal{C}_\beta^{(n)}$ are the nth roots of $\overline{\beta}$.*

In other words, if the mass point is inserted somewhere where there would already be an eigenvalue with only the Lebesgue measure, then none of the eigenvalues are changed.

Proof. As proven above,

$$\Phi_{n-1}(z) = z^{n-1} - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=0}^{n-2}\omega^{(n-1)-k}z^k$$

and hence

$$\Phi_{n-1}^*(z) = 1 - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\overline{\omega^k}z^k$$

which gives

$$\begin{aligned}\Phi_n(z) &= z^n - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\omega^{n-k}z^k - \frac{\overline{\beta}\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\overline{\omega^k}z^k - \overline{\beta} \\ &= z^n - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\omega^{n-k}z^k - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\overline{\beta}\overline{\omega^k}z^k - \overline{\beta}.\end{aligned}$$

Knowing that $\overline{\beta} = \omega^n$, we get

$$\begin{aligned}\Phi_n(z) &= z^n - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\omega^{n-k}z^k - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\omega^n\overline{\omega^k}z^k - \overline{\beta} \\ &= z^n - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\omega^{n-k}z^k - \frac{\gamma}{1+n\gamma-2\gamma}\sum_{k=1}^{n-1}\omega^{n-k}z^k - \overline{\beta} \\ &= z^n - \overline{\beta}\end{aligned}$$

since $\overline{\omega} = \omega^{-1}$.

Hence, since the roots of the paraorthogonal polynomial are the eigenvalues of the finite matrix, if λ is an eigenvalue of $\mathcal{C}_\beta^{(n)}$, then $\lambda^n = \overline{\beta}$. □

3 Rotation of Measures

3.1 Results

In this section, we will be interested in the relationship between a CMV matrix and a CMV matrix associated with the same measure, but rotated by a certain angle. Specifically, we would like to show a simple relationship between the Verblunsky coefficients of the two matrices. We will begin by investigating the relationship between the moments of the two measures.

Lemma 3.1. *Let μ be a measure on the unit circle and let $\widetilde{\mu}$ be a measure defined by $\widetilde{\mu}(A) = \mu(e^{-i\theta_0}A)$ for any measurable set A. Furthermore, let c_n be the moments of $d\mu$ and $\widetilde{c}_n$ the moments of $d\widetilde{\mu}$. Then*

$$\widetilde{c}_n = e^{-in\theta_0}c_n. \tag{3.1}$$

Proof. We know that:

$$\begin{aligned}\widetilde{c}_n &= \int_{\partial\mathbb{D}} z^{-n}d\widetilde{\mu} \\ &= \int_{\partial\mathbb{D}} e^{-in\theta}d\widetilde{\mu}(\theta) \\ &= \int_{\partial\mathbb{D}} e^{-in\theta_0}e^{-in(\theta-\theta_0)}d\mu(\theta-\theta_0) \\ &= e^{-in\theta_0}\int_{\partial\mathbb{D}} e^{-in(\theta-\theta_0)}d\mu(\theta-\theta_0) \\ &= e^{-in\theta_0}c_n\end{aligned}$$

by definition of c_n. □

Now we will move onto the orthogonal polynomials.

Lemma 3.2. *Let μ and $\widetilde{\mu}$ be defined as in Lemma 3.1 and let $\Phi_n(z)$ and $\varphi_n(z)$ be the monic orthogonal and orthonormal polynomials respectively for $d\mu$ and $\widetilde{\Phi}_n(z)$ and $\widetilde{\varphi}_n(z)$ the same for $d\widetilde{\mu}$. If $a_k^{(n)}$, $b_k^{(n)}$, $\widetilde{a}_k^{(n)}$, and $\widetilde{b}_k^{(n)}$ are defined by*

$$\begin{aligned}\Phi_n(z) &= a_0^{(n)} + a_1^{(n)} z + \cdots + a_n^{(n)} z^n \\ \widetilde{\Phi}_n(z) &= \widetilde{a}_0^{(n)} + \widetilde{a}_1^{(n)} z + \cdots + \widetilde{a}_n^{(n)} z^n \\ \varphi_n(z) &= b_0^{(n)} + b_1^{(n)} z + \cdots + b_n^{(n)} z^n \\ \widetilde{\varphi}_n(z) &= \widetilde{b}_0^{(n)} + \widetilde{b}_1^{(n)} z + \cdots + \widetilde{b}_n^{(n)} z^n\end{aligned}$$

Then we have that

$$\widetilde{a}_k^{(n)} = e^{i(n-k)\theta_0} a_k^{(n)} \tag{3.2}$$

and

$$\widetilde{b}_k^{(n)} = e^{i(n-k)\theta_0} b_k^{(n)}. \tag{3.3}$$

Proof. We will proceed by induction. Since $\Phi_0(z) = \widetilde{\Phi}_0(z) = \varphi_0(z) = \widetilde{\varphi}_0(z) = 1$, the base case is trivial.

Let us now calculate the relationship between the coefficients of the monic orthogonal polynomials and the preceding orthonormal polynomials. We know from equation (1.3) that

$$\begin{aligned}\Phi_n(z) &= z^n - \sum_{j=0}^{n-1} \langle \varphi_j(z), z^n \rangle \varphi_j \\ &= z^n - \sum_{j=0}^{n-1} \langle \varphi_j(z), z^n \rangle \sum_{k=0}^{j} b_k^{(j)} z^k.\end{aligned}$$

Reversing the order of summation, we get

$$\Phi_n(z) = z^n - \sum_{k=0}^{n-1} z^k \sum_{j=k}^{n-1} \langle \varphi_j(z), z^n \rangle b_k^{(j)}$$

and hence

$$a_k^{(n)} = \sum_{j=k}^{n-1} \langle \varphi_j(z), z^n \rangle b_k^{(j)}. \tag{3.4}$$

This expression for $a_k^{(n)}$ is equally valid for $\widetilde{a}_k^{(n)}$ if we replace everything by its analog for $d\widetilde{\mu}$. Hence if $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ is the inner product in $L^2(\partial\mathbb{D}, d\widetilde{\mu})$, then

$$\widetilde{a}_k^{(n)} = \sum_{j=k}^{n-1} \langle\!\langle \widetilde{\varphi}_j(z), z^n \rangle\!\rangle \widetilde{b}_k^{(j)}.$$

Examining $\langle\!\langle \widetilde{\varphi}_j(z), z^n \rangle\!\rangle$, we find that for $0 \leq j \leq n-1$

$$\begin{aligned}
\langle\!\langle \widetilde{\varphi}_j(z), z^n \rangle\!\rangle &= \langle\!\langle \sum_{l=0}^{j} \widetilde{b}_l^{(j)} z^l, z^n \rangle\!\rangle \\
&= \langle\!\langle \sum_{l=0}^{j} e^{i(j-l)\theta_0} b_l^{(j)} z^l, z^n \rangle\!\rangle \\
&= \sum_{l=0}^{j} e^{-i(j-l)\theta_0} \overline{b}_l^{(j)} \langle\!\langle z^l, z^n \rangle\!\rangle \\
&= \sum_{l=0}^{j} e^{-i(j-l)\theta_0} \overline{b}_l^{(j)} \int_{\partial\mathbb{D}} z^{n-l} d\widetilde{\mu} \\
&= \sum_{l=0}^{j} e^{-i(j-l)\theta_0} \overline{b}_l^{(j)} \widetilde{c}_{l-n} \\
&= \sum_{l=0}^{j} e^{-i(j-l)\theta_0} \overline{b}_l^{(j)} e^{-i(l-n)\theta_0} c_{l-n} \\
&= \sum_{l=0}^{j} e^{-i(j-n)\theta_0} \overline{b}_l^{(j)} c_{l-n} \\
&= e^{i(n-j)\theta_0} \sum_{l=0}^{j} \overline{b}_l^{(j)} \langle z^l, z^n \rangle \\
&= e^{i(n-j)\theta_0} \langle \sum_{l=0}^{j} b_l^{(j)} z^l, z^n \rangle \\
&= e^{i(n-j)\theta_0} \langle \varphi_j(z), z^n \rangle
\end{aligned}$$

by the induction hypothesis and equation (3.1).

With this new expression for $\langle\!\langle \widetilde{\varphi}_j(z), z^n \rangle\!\rangle$ and the induction hypothesis, we now find that

$$\begin{aligned}
\widetilde{a}_k^{(n)} &= \sum_{j=k}^{n-1} \langle\!\langle \widetilde{\varphi}_j(z), z^n \rangle\!\rangle \widetilde{b}_k^{(j)} \\
&= \sum_{j=k}^{n-1} e^{i(n-j)\theta_0} \langle \varphi_j(z), z^n \rangle e^{i(j-k)\theta_0} b_k^{(j)} \\
&= \sum_{j=k}^{n-1} e^{i(n-k)\theta_0} \langle \varphi_j(z), z^n \rangle b_k^{(j)} \\
&= e^{i(n-k)\theta_0} \sum_{j=k}^{n-1} \langle \varphi_j(z), z^n \rangle b_k^{(j)} \\
&= e^{i(n-k)\theta_0} a_k^{(n)}
\end{aligned}$$

by equation (3.4). Hence equation (3.2) holds; we must simply prove that this implies that equation (3.3) holds.

We know from (1.13) that

$$\widetilde{\varphi}_n(z) = \frac{\widetilde{\Phi}_n(z)}{\widetilde{\rho}_0 \widetilde{\rho}_1 \cdots \widetilde{\rho}_{n-1}}$$

but $\widetilde{\rho}_j = \sqrt{1 - |\widetilde{\alpha}_j|^2}$, equation (1.12), and equation (3.2) imply that

$$\begin{aligned}
\widetilde{\alpha}_j &= -\overline{\widetilde{\Phi}_{j+1}(0)} \\
&= -\overline{e^{i(j+1)\theta_0} \Phi_{j+1}(0)} \\
&= e^{-i(j+1)\theta_0} \alpha_j.
\end{aligned}$$

This holds by the induction hypothesis for $0 \leq j \leq n-2$, and by the above proof for $j = n-1$. So

$$\begin{aligned}
\widetilde{\rho}_j &= \sqrt{1 - |\widetilde{\alpha}_j|^2} \\
&= \sqrt{1 - |e^{-i(j+1)\theta_0} \alpha_j|^2} \\
&= \sqrt{1 - |\alpha_j|^2} \\
&= \rho_j
\end{aligned}$$

for $0 \leq j \leq n-1$. Thus

$$\widetilde{\varphi}_n(z) = \frac{\widetilde{\Phi}_n(z)}{\rho_0 \rho_1 \cdots \rho_{n-1}}$$

and hence

$$\begin{aligned}
\widetilde{b}_k^{(n)} &= \frac{\widetilde{a}_k^{(n)}}{\rho_0\rho_1\cdots\rho_{n-1}} \\
&= \frac{e^{i(n-k)\theta_0} a_k^{(n)}}{\rho_0\rho_1\cdots\rho_{n-1}} \\
&= e^{i(n-k)\theta_0}\frac{a_k^{(n)}}{\rho_0\rho_1\cdots\rho_{n-1}} \\
&= e^{i(n-k)\theta_0} b_k^{(n)}
\end{aligned}$$

as desired. □

Next we must investigate the behavior of the paraorthogonal polynomials if we are to deal with finite matrices.

Lemma 3.3. *Let $\Phi_{n,\beta}$ be the nth degree paraorthogonal polynomial associated with $d\mu$ with $\beta \in \partial\mathbb{D}$ the last inserted Verblunsky coefficient and let $\widetilde{\Phi}_{n,e^{-in\theta_0}\beta}$ be the nth degree paraorthogonal polynomial associated with $d\widetilde{\mu}$. Then if*

$$\Phi_n = a_0 + a_1 z + \cdots + a_n z^n$$

and

$$\widetilde{\Phi}_n = \widetilde{a}_0 + \widetilde{a}_1 z + \cdots + \widetilde{a}_n z^n$$

then

$$\widetilde{a}_k = e^{i(n-k)\theta_0} a_k \tag{3.5}$$

Proof. We know that:

$$\begin{aligned}
\Phi_n(z) &= z\Phi_{n-1}(z) - \overline{\beta}\Phi_{n-1}^*(z) \\
&= z\sum_{k=0}^{n-1}\left(a_k^{(n-1)} z^k\right) - \overline{\beta}\sum_{k=0}^{n-1}\left(\overline{a_{n-1-k}^{(n-1)}} z^k\right) \\
&= z^n + \sum_{k=1}^{n-1}\left[\left(a_{k-1}^{(n-1)} - \overline{\beta a_{n-1-k}^{(n-1)}}\right) z^k\right] - \overline{\beta a_{n-1}^{(n-1)}}
\end{aligned}$$

so by Lemma 3.2

$$
\begin{aligned}
\widetilde{\Phi}_n(z) &= z\widetilde{\Phi}_{n-1}(z) - \overline{e^{-in\theta_0}\beta}\widetilde{\Phi}^*_{n-1}(z) \\
&= z\sum_{k=0}^{n-1}\left(\widetilde{a}_k^{(n-1)}z^k\right) - e^{in\theta_0}\overline{\beta}\sum_{k=0}^{n-1}\left(\overline{\widetilde{a}_{n-1-k}^{(n-1)}}z^k\right) \\
&= \sum_{k=0}^{n-1}\left(e^{i(n-1-k)\theta_0}a_k^{(n-1)}z^{k+1}\right) - e^{in\theta_0}\overline{\beta}\sum_{k=0}^{n-1}\left(e^{-ik\theta_0}\overline{a_{n-1-k}^{(n-1)}}z^k\right) \\
&= \sum_{k=1}^{n}\left(e^{i(n-k)\theta_0}a_{k-1}^{(n-1)}z^k\right) - \sum_{k=0}^{n-1}\left(e^{i(n-k)\theta}\overline{\beta a_{n-1-k}^{(n-1)}}z^k\right) \\
&= z^n + \sum_{k=1}^{n-1}\left[e^{i(n-k)\theta_0}\left(a_{k-1}^{(n-1)} - \overline{\beta a_{n-1-k}^{(n-1)}}\right)z^k\right] - e^{in\theta_0}\overline{\beta a_{n-1}^{(n-1)}} \\
&= z^n + \sum_{k=1}^{n-1}\left[e^{i(n-k)\theta_0}a_k z^k\right] - e^{i(n-0)\theta_0}a_0.
\end{aligned}
$$

It follows that equation (3.5) holds. □

Hence the same relationship holds between the coefficients of the paraorthogonal polynomials as for the orthogonal polynomials.

And we finally have enough information to prove the following relationship:

Theorem 3.4. *Let μ and $\widetilde{\mu}$ be defined as in Lemma 3.1. Let α_n be the Verblunsky coefficients for $d\mu$ and $\widetilde{\alpha}_n$ those for $d\widetilde{\mu}$ and let $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots)$ and $\widetilde{\mathcal{C}} = \mathcal{C}(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \ldots)$. Then*

1. *$\widetilde{\alpha}_n = e^{-i(n+1)\theta_0}\alpha_n$.*
2. *If λ is an eigenvalue of $\mathcal{C}^{(n)}$, then $e^{i\theta_0}\lambda$ is an eigenvalue of $\widetilde{\mathcal{C}}^{(n)}$.*
3. *If λ is an eigenvalue of $\mathcal{C}_\beta^{(n)}$, then $e^{i\theta_0}\lambda$ is an eigenvalue of $\widetilde{\mathcal{C}}^{(n)}_{e^{-in\theta_0}\beta}$.*

Proof. (1) follows from Lemma 3.2 by (1.12).

To prove (2), we must use the fact from [6] that the eigenvalues of $\mathcal{C}^{(n)}$ are exactly the zeroes of $\Phi_n(z)$. So if λ is an eigenvalue of $\mathcal{C}^{(n)}$, then

$$
\begin{aligned}
\Phi_n(\lambda) &= a_0^{(n)} + a_1^{(n)}\lambda + \cdots + a_n^{(n)}\lambda^n \\
&= \sum_{k=0}^{n} a_k^{(n)}\lambda^k \\
&= 0.
\end{aligned}
$$

Thus by Lemma 3.2:

$$\begin{aligned}
\widetilde{\Phi}_n(e^{i\theta_0}\lambda) &= \sum_{k=0}^{n} \widetilde{a}_k^{(n)} e^{ik\theta_0}\lambda^k \\
&= \sum_{k=0}^{n} e^{i(n-k)\theta_0} a_k^{(n)} e^{ik\theta_0}\lambda^k \\
&= \sum_{k=0}^{n} e^{in\theta_0} a_k^{(n)} \lambda^k \\
&= e^{in\theta_0} \sum_{k=0}^{n} a_k^{(n)} \lambda^k \\
&= 0.
\end{aligned}$$

Hence $e^{i\theta_0}\lambda$ is an eigenvalue of $\widetilde{\mathcal{C}}^{(n)}$.

The proof of (3) is nearly identical to that of (2) , but we use Lemma 3.3 to show the relationship between the coefficients of the paraorthogonal polynomials. □

3.2 Applications

We will now examine some consequences of the measure rotation formulas derived in the previous section. We will begin with a useful definition.

Definition 3.5. *We say that a measure μ on the unit circle is* periodic *with period θ_0 if $\mu(A) = \mu(e^{-i\theta_0}A)$ for all measurable sets A.*

And now we can prove an interesting corollary.

Corollary 3.6. *Let μ be a measure on the unit circle with moments $\{c_n\}_{n\geq 0}$ and Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ and let $k \in \mathbb{Z}^+$. Then the following are equivalent:*

1. *μ is periodic with period $2\pi/k$.*
2. *$c_n = 0\ \forall n \not\equiv 0 \pmod{k}$.*
3. *$\alpha_n = 0\ \forall n \not\equiv -1 \pmod{k}$.*

Proof. (1) ⇒ (2): Let μ be a periodic measure on the unit circle with period period $2\pi/k$ for some $k \in \mathbb{Z}^+$ and define $\widetilde{\mu}$ by $\widetilde{\mu}(A) = \mu(e^{-2i\pi/k}A)$ for all measurable sets A. By definition of periodic, $\widetilde{\mu} = \mu$. Thus we know from Lemma 3.1 that

$$c_n = e^{-2in\pi/k} c_n.$$

Thus, if $n \not\equiv 0 \pmod{k}$, then $\frac{n}{k} \notin \mathbb{Z}$. Hence $e^{-2in\pi/k} \neq 1$. Consequently

$$(1 - e^{-2in\pi/k})c_n = 0$$

implies that $c_n = 0 \ \forall \ n \not\equiv 0 \pmod{k}$, as desired.

(2) $\Rightarrow$ (1): Let $\{c_n\}_{n\geq 0}$ be moments such that $c_n = 0 \ \forall n \not\equiv 0 \pmod{k}$ for some $k \in \mathbb{Z}^+$. This defines a measure μ. We now define a new set of moments by $\{e^{-2in\pi/k}c_n\}_{n\geq 0}$. By Lemma 3.1, this defines a measure $\widetilde{\mu}$ such that for any measurable set A $\widetilde{\mu}(A) = \mu(e^{-2i\pi/k}A)$. But $\forall n \equiv 0 \pmod{k}$, we have $\frac{n}{k} \in \mathbb{Z}$. Hence $e^{-2in\pi/k} = 1$. Thus $\{c_n\}_{n\geq 0} = \{e^{-2in\pi/k}c_n\}_{n\geq 0}$. This implies that $\widetilde{\mu} = \mu$. So $\mu(A) = \mu(e^{-2i\pi/k}A)$ for any measurable set A. Thus μ is periodic with period $2\pi/k$.

(1) $\Rightarrow$ (3): Let μ be a periodic measure on the unit circle with period $2\pi/k$ for some $k \in \mathbb{Z}^+$ and define $\widetilde{\mu}$ by $\widetilde{\mu}(A) = \mu(e^{-2i\pi/k}A)$ for all measurable sets A. By definition of periodic, $\widetilde{\mu} = \mu$. Thus we know from Theorem 3.4 that

$$\alpha_n = e^{-2i(n+1)\pi/k}\alpha_n$$

Consequently, if $n \not\equiv -1 \pmod{k}$, then $\frac{n+1}{k} \notin \mathbb{Z}$. Thus $e^{-2i(n+1)\pi/k} \neq 1$. Therefore

$$(1 - e^{-2i(n+1)\pi/k})\alpha_n = 0$$

implies that $\alpha_n = 0 \ \forall n \not\equiv -1 \pmod{k}$, as desired.

(3) $\Rightarrow$ (1): Let $\{\alpha_n\}_{n\geq 0}$ be Verblunsky coefficients such that $\alpha_n = 0$ $\forall n \not\equiv -1 \pmod{k}$ for some $k \in \mathbb{Z}^+$. This defines a measure μ, the spectral measure of $\mathcal{C}(\alpha_0, \alpha_1, \cdots)$. If we define a new set of Verblunsky coefficients by $\{e^{-2i(n+1)\pi/k}\alpha_n\}_{n\geq 0}$. By Theorem 3.4, this defines another measure $\widetilde{\mu}$ such that for any measurable set A $\widetilde{\mu}(A) = \mu(e^{-2i\pi/k}A)$. However we know that $e^{-2i(n+1)\pi/k} = 1$ if $\frac{n+1}{k} \in \mathbb{Z}$. Hence $e^{-2i(n+1)\pi/k} = 1 \ \forall n \equiv -1 \pmod{k}$. Thus, since $\alpha_n = 0 \ \forall n \not\equiv -1 \pmod{k}$, $\{\alpha_n\}_{n\geq 0} = \{e^{-2i(n+1)\pi/k}\alpha_n\}_{n\geq 0}$. Therefore we know that $\mu = \widetilde{\mu}$ and it follows that $\mu(A) = \mu(e^{-2i\pi/k}A)$ for any measurable set A. Hence μ is periodic with period $2\pi/k$. $\square$

It is interesting to note that this also provides an alternate way to calculate the Verblunsky coefficients for λ, the Lebesgue measure.

Example 3.7. We know that for all measurable sets A, $\lambda(A) = \lambda(e^{-2i\pi/k}A)$ for any $k \in \mathbb{Z}^+$. Hence by Corollary 3.6, $\alpha_n = 0 \; \forall n \not\equiv -1 \pmod{k} \; \forall k \in \mathbb{Z}^+$. But this means that $\alpha_n = 0 \; \forall n \geq 0$.

4 Clock Behavior of Single Inserted Mass Point

In this section, we will be interested in the question of what the eigenvalues of finite truncations of matrices associated with the single inserted mass point are more generally than stated in Theorem 2.3. The tools developed in Section 3 will allow us to look at only a single measure due to the fact that all of the measures generated by inserting a mass point into the Lebesgue measure are rotations of each other. This will significantly ease calculations.

4.1 Motivation

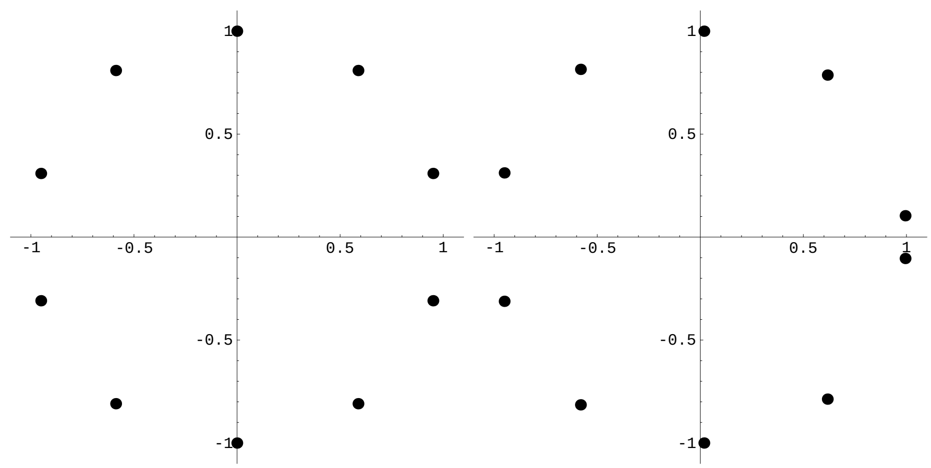
As seen in Theorem 2.3, it is possible for the eigenvalues of a finite matrix associated with a single inserted mass point measure to be exactly the same as those of finite matrices associated with the Lebesgue measure. But that naturally raises the question, "What if those special conditions are not met?" So let us look at some graphs to see what we expect to happen.

Let $\mu_0 = (1-\gamma)\lambda + \gamma\delta_1$ and let $S = \mathcal{C}(d\mu_0)$. Then we know from Theorem 2.3 that the eigenvalues of $S_1^{(n)}$ are the nth roots of unity regardless of the value of γ. So in order to determine what to expect in the very opposite case, let us consider $S_{-1}^{(n)}$. Figure 4.1 shows the similarities that most of the eigenvalues of $S_{-1}^{(n)}$ have for one specific case. As we can see, only the eigenvalues which are closest to the inserted mass are significantly perturbed from the eigenvalues of the free matrix counterpart.

In general, only the eigenvalues which are closest to the mass point show significant perturbation from the expected clock points. We would like to determine exactly how far away from the clock points the eigenvalues are and, if possible, show exactly the relationship with the clock points as n grows without bound.

4.2 Construction of Eigenvectors

In order to determine the exact behavior of the eigenvalues of the finite matrices associated with μ, we will have to examine the eigenvectors to which they are associated. When looking at the finite matrix, it is not easy to determine what the eigenvectors are. However, when viewed as an



(a) Eigenvalues of $\Lambda_{-1}^{(20)}$ (b) Eigenvalues of $S_{-1}^{(20)}$ for $\gamma = \frac{1}{2}$

Figure 1: Compared Eigenvalues

operator on $L^2(\partial\mathbb{D}, d\mu)$ with the proper measure (the spectral measure of the matrix), the operator corresponds simply to multiplication by z, and eigenfunctions are much easier to determine.

First, let us prove the following lemma:

Lemma 4.1. *Let $\mu = (1-\gamma)\lambda + \gamma\delta_1$ and let $S = \mathcal{C}(d\mu)$. Now let $\mu_\beta^{(n)}$ be the spectral measure of $S_\beta^{(n)}$. Then the paraorthognal polynomial $\Phi_{n,\beta}(z) = 0$ almost everywhere in $L^2(\partial\mathbb{D}, d\mu_\beta^{(n)})$.*

Proof. We know from [4] that $\Phi_{n,\beta}(z)$ is the characteristic polynomial of $S_\beta^{(n)}$. This implies that $\Phi_{n,\beta}(S_\beta^{(n)}) = 0$. Define $M_z : L^2(\partial\mathbb{D}, d\mu_\beta^{(n)}) \to L^2(\partial\mathbb{D}, d\mu_\beta^{(n)})$ by $M_z : f(z) \mapsto zf(z)$. By the spectral theorem, we know that $\Phi_{n,\beta}(M_z) = 0$. We know, however, that $\Phi_{n,\beta}(M_z)(f(z)) = \Phi_{n,\beta}(z)f(z) = 0$ $\forall f \in L^2(\partial\mathbb{D}, d\mu_\beta^{(n)})$. Hence $\Phi_{n,\beta}(z) = 0$ almost everywhere. □

This will give us the tools that we need in order to find the form of the eigenvectors.

Lemma 4.2. *Let $\Phi_{n,\beta}(z) = \sum_{j=0}^{n} a_j z^j$ and let λ be any zero of $\Phi_{n,\beta}(z)$. If we define $f(z) = \sum_{k=0}^{n-1} b_k(\lambda) z^k$ where*

$$b_j(\lambda) = -\frac{1}{\lambda^{j+1}} \left(\sum_{k=0}^{j} a_k \lambda^k \right) \qquad \text{for } 0 \le j \le n-2$$

and $b_{n-1}(\lambda) = 1$*, then* $M_z(f(z)) = \lambda f(z)$ *in* $L^2(\partial\mathbb{D}, d\mu_\beta^{(n)})$.

Proof. Let us determine what the form of $M_z(f(z))$ is. Because $\Phi_{n,\beta}$ is monic, we know that $a_n = 1$. Hence

$$\begin{aligned} M_z(f(z)) &= zf(z) \\ &= z\sum_{j=0}^{n-1} b_j z^j \\ &= b_{n-1}(\lambda)z^n + \sum_{j=0}^{n-2} b_j(\lambda)z^{j+1} \\ &= z^n + \sum_{j=0}^{n-2} b_j(\lambda)z^{j+1} \\ &= z^n - \Phi_{n,\beta} + \sum_{j=0}^{n-2} b_j(\lambda)z^{j+1} \\ &= -\sum_{j=0}^{n-2} a_j z^j + \sum_{j=0}^{n-2} b_j(\lambda)z^{j+1} \\ &= \sum_{j=1}^{n-1} \left[(b_{j-1}(\lambda) - a_j)\, z^j\right] - a_0. \end{aligned}$$

We know that

$$\begin{aligned} -a_0 &= \lambda(-\frac{1}{\lambda}(a_0)) \\ &= \lambda b_0(\lambda). \end{aligned}$$

Furthermore if $1 \leq j \leq n-2$

$$\begin{aligned} b_{j-1}(\lambda) - a_j &= -\frac{1}{\lambda^j}\left(\sum_{k=0}^{j-1} a_k\lambda^k\right) - a_j \\ &= -\frac{1}{\lambda^j}\left(\sum_{k=0}^{j} a_k\lambda^k\right) \\ &= \lambda\left(-\frac{1}{\lambda^{j+1}}\left(\sum_{k=0}^{j} a_k\lambda^k\right)\right) \\ &= \lambda b_j(\lambda). \end{aligned}$$

Finally

$$
\begin{aligned}
b_{n-2}(\lambda) - a_{n-1} &= -\frac{1}{\lambda^{n-1}} \left(\sum_{k=0}^{n-2} a_k \lambda^k \right) - a_{n-1} \\
&= -\frac{1}{\lambda^{n-1}} \left(\sum_{k=0}^{n-1} a_k \lambda^k \right) \\
&= -\frac{1}{\lambda^{n-1}} \left(\Phi_{n,\beta}(\lambda) - \lambda^n \right) \\
&= -\frac{1}{\lambda^{n-1}} (-\lambda^n) \\
&= \lambda \\
&= \lambda b_{n-1}(\lambda)
\end{aligned}
$$

Thus we have shown that $M_z(f(z)) = \lambda f(z)$. □

Clearly it is possible to find an eigenvector such that $b_0(\lambda) = 1$ as well. Thus we have the following lemma as well.

Lemma 4.3. *Let $\Phi_{n,\beta}(z) = \sum_{j=0}^{n} a_j z^j$ and let λ be any zero of $\Phi_{n,\beta}(z)$. If we define $f(z) = \sum_{j=0}^{n-1} b_k(\lambda) z^k$ where*

$$
b_j(\lambda) = -\frac{\lambda}{a_0} \left(\sum_{k=j}^{n} a_k \lambda^{k-j} \right) \qquad \text{for } 1 \leq j \leq n-1
$$

and $b_0(\lambda) = 1$, then $M_z(f(z)) = \lambda f(z)$.

The proof of this is similar to the proof of Lemma 4.2 and will therefore be omitted here.

4.3 Change of Basis

It will frequently be useful to consider changing basis from $\{1, z, z^{-1}, z^2, \ldots\}$ to $\{\chi_0, \chi_1, \chi_2, \ldots\}$ since it is with respect to this orthonormal basis that a CMV matrix is the representation of multiplication by z.

Since $\{\chi_0, \chi_1, \chi_2, \ldots\}$ is an orthonormal basis, we can easily write the matrix for a change of basis into this basis in terms of inner products. Specifically, the matrix representation of a change of basis from $\{1, z, z^{-1}, z^2, \ldots\}$

is

$$\begin{pmatrix} \langle \chi_0, 1\rangle & \langle \chi_0, z\rangle & \langle \chi_0, z^{-1}\rangle & \cdots \\ \langle \chi_1, 1\rangle & \langle \chi_1, z\rangle & \langle \chi_1, z^{-1}\rangle & \cdots \\ \langle \chi_2, 1\rangle & \langle \chi_2, z\rangle & \langle \chi_2, z^{-1}\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Because of the definition of χ_n, we know that this matrix is actually upper triangular. So it is given by

$$\begin{pmatrix} \langle \chi_0, 1\rangle & \langle \chi_0, z\rangle & \langle \chi_0, z^{-1}\rangle & \langle \chi_0, z^2\rangle & \cdots \\ 0 & \langle \chi_1, z\rangle & \langle \chi_1, z^{-1}\rangle & \langle \chi_1, z^2\rangle & \cdots \\ 0 & 0 & \langle \chi_2, z^{-1}\rangle & \langle \chi_2, z^2\rangle & \cdots \\ 0 & 0 & 0 & \langle \chi_3, z^2\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

One final refinement that we can make to our understanding of this matrix is to look at the terms on the main diagonal. We must consider even and odd cases. If $n = 2k$, then we have

$$\begin{aligned} \langle \chi_{2k}, z^{-k}\rangle &= \langle z^{-k}\varphi^*_{2k}, z^{-k}\rangle \\ &= \langle \varphi^*_{2k}, 1\rangle \\ &= \langle z^{2k}, \varphi_{2k}\rangle \\ &= \frac{1}{\|\Phi_{2k}\|}\langle z^{2k}, \Phi_{2k}\rangle \\ &= \frac{1}{\|\Phi_{2k}\|}\langle z^{2k}, P(z^{2k})\rangle \end{aligned}$$

where P is the orthogonal projection onto $\mathrm{Span}\{1, z, z^2, \ldots, z^{2k-1}\}^\perp$. Since projections have the property that $P = P^* = P^2$, we have the following:

$$\begin{aligned} \langle \chi_{2k}, z^{-k}\rangle &= \frac{1}{\|\Phi_{2k}\|}\langle z^{2k}, P(z^{2k})\rangle \\ &= \frac{1}{\|\Phi_{2k}\|}\langle P(z^{2k}), P(z^{2k})\rangle \\ &= \frac{1}{\|\Phi_{2k}\|}\langle \Phi_{2k}, \Phi_{2k}\rangle \\ &= \|\Phi_{2k}\| \end{aligned}$$

A similar calculation shows that

$$\langle \chi_{2k-1}, z^k\rangle = \|\Phi_{2k-1}\|$$

Hence the matrix form of the change of basis is

$$\begin{pmatrix} \|\Phi_0\| & \langle \chi_0, z \rangle & \langle \chi_0, z^{-1} \rangle & \langle \chi_0, z^2 \rangle & \cdots \\ 0 & \|\Phi_1\| & \langle \chi_1, z^{-1} \rangle & \langle \chi_1, z^2 \rangle & \cdots \\ 0 & 0 & \|\Phi_2\| & \langle \chi_2, z^2 \rangle & \cdots \\ 0 & 0 & 0 & \|\Phi_3\| & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

4.4 Eigenvector Approximations

While the formula for the eigenvectors presented in Section 4.2 is of little use unless the eigenvalues are already known, a small modification provides useful results.

Lemmas 4.2 and 4.3 provide formulas for the eigenvalues as functions of λ, the zeroes of the paraorthogonal polynomial. A natrual question which arises from this formula is the question of what happens when the λ used in the formula is not a zero of the paraorthogonal polynomial. If λ is changed only slightly from an eigenvalue, we expect to obtain something which is very close to an eigenvector, even though it is clearly not actually an eigenvector. The exploration of this question leads to the following lemma.

Lemma 4.4. *Let* $\Phi_{n,\beta} = \sum_{k=0}^{n} a_j z^j$ *be a paraorthogonal polynomial and let* $\lambda \in \partial\mathbb{D}$. *If* $f(z) = \sum_{k=0}^{n-1} b_k(\lambda) z^k$ *is defined as in Lemma 4.2 or 4.3, then*

$$\|(M_z - \lambda)(f(z))\| = |\Phi_{n,\beta}(\lambda)|$$

Once again, we will prove this for coefficients as defined in Lemma 4.2 and omit the proof for coefficients as defined in Lemma 4.3 as the proofs are very similar.

Proof. As shown in Lemma 4.2,

$$\begin{aligned} M_z(f(z)) &= zf(z) \\ &= \sum_{k=1}^{n-1} (b_{k-1} - a_k) - a_0. \end{aligned}$$

Hence we have

$$\begin{aligned} (M_z - \lambda)(f(z)) &= \sum_{k=1}^{n-1} (b_{k-1} - a_k - \lambda b_k) - (a_0 + \lambda b_0) \\ &= \sum_{k=1}^{n-1} (b_{k-1} - a_k - \lambda b_k) \end{aligned}$$

since $b_0 = -\frac{a_0}{\lambda}$. Examining the other terms, we get for $1 \leq k \leq n-2$

$$\begin{aligned} b_{k-1} - a_k - \lambda b_k &= -\frac{1}{\lambda^k}\left(\sum_{j=0}^{k-1} a_k \lambda^k\right) - a_k - \lambda b_k \\ &= -\frac{1}{\lambda^k}\left(\sum_{j=0}^{k} a_k \lambda^k\right) - \lambda b_k \\ &= \lambda b_k - \lambda b_k \\ &= 0. \end{aligned}$$

Finally

$$\begin{aligned} b_{n-2} - a_{n-1} - \lambda b_{n-1} &= b_{n-2} - a_{n-1} - \lambda \\ &= -\frac{1}{\lambda^{n-1}}\left(\sum_{k=0}^{n-2} a_k \lambda^k\right) - a_{n-1} - \lambda \\ &= -\frac{1}{\lambda^{n-1}}\left(\sum_{k=0}^{n-1} a_k \lambda^k\right) - \lambda a_n \\ &= -\frac{1}{\lambda^{n-1}}\left(\sum_{k=0}^{n} a_k \lambda^k\right) \\ &= -\frac{1}{\lambda^{n-1}}\Phi_{n,\beta}(\lambda) \end{aligned}$$

Hence

$$(M_z - \lambda)(f(z)) = -\frac{1}{\lambda^{n-1}}\Phi_{n,\beta}(\lambda) z^{n-1}.$$

Thus

$$\|(M_z - \lambda)(f(z))\| = |\Phi_{n,\beta}(\lambda)|$$

as desired. □

This is only one step on the way to finding eigenvalues of finite matrices due to the fact that we know nothing about the norm of $f(z)$. But it will be enough to tell us something about the eigenvalues of the case when we are looking at the single inserted mass point measure.

4.5 Single Inserted Mass Point

It is at this point that we will turn to a specific example. Once again we will let $\mu = (1-\gamma)\lambda + \gamma\delta_1$ and $S = \mathcal{C}(d\mu)$. First we will want to determine

$\Phi_{n,\beta}$. Theorem 2.2 gives us

$$\Phi_{n-1}(z) = z^{n-1} - \frac{\gamma}{1+n\gamma-2\gamma}\left(\sum_{k=0}^{n-2} z^k\right).$$

Hence

$$\Phi_{n-1}^*(z) = 1 - \frac{\gamma}{1+n\gamma-2\gamma}\left(\sum_{k=1}^{n-1} z^k\right).$$

Therefore we conclude that

$$\begin{aligned}\Phi_{n,\beta}(z) &= z\Phi_{n-1} - \overline{\beta}\Phi_{n-1}^* \\ &= z^n - \frac{\gamma}{1+n\gamma-2\gamma}\left(\sum_{k=1}^{n-1} z^k\right) + \frac{\overline{\beta}\gamma}{1+n\gamma-2\gamma}\left(\sum_{k=1}^{n-1} z^k\right) - \overline{\beta} \\ &= z^n + \frac{\gamma(\overline{\beta}-1)}{1+n\gamma-2\gamma}\left(\sum_{k=1}^{n-1} z^k\right) - \overline{\beta}.\end{aligned}$$

Now it is our guess that the zeroes of this polynomial are close to nth roots of $\overline{\beta}$. So let $\lambda \in \partial\mathbb{D}$ such that $\lambda^n = \overline{\beta}$. Then we have

$$\begin{aligned}\Phi_{n,\beta}(\lambda) &= \lambda^n - \frac{\gamma(\overline{\beta}-1)}{1+n\gamma-2\gamma}\left(\sum_{k=1}^{n-1} \lambda^k\right) - \overline{\beta} \\ &= \frac{\gamma(\overline{\beta}-1)}{1+n\gamma-2\gamma}\left(\sum_{k=1}^{n-1} \lambda^k\right) \\ &= \frac{\gamma(\overline{\beta}-1)}{1+n\gamma-2\gamma}\left(\frac{\lambda^n-\lambda}{\lambda-1}\right).\end{aligned}$$

Knowing that $|\lambda^n - \lambda| \leq 2$ and $|\overline{\beta}-1| \leq 2$ since $\lambda, \beta \in \partial\mathbb{D}$, we thus have that

$$|\Phi_{n,\beta}(\lambda)| \leq \frac{4\gamma}{(\lambda-1)(1+n\gamma-2\gamma)}. \tag{4.1}$$

We will now wish to find an estimate for how far away from an eigenvalue this is. We have

$$\|(M_z - \lambda)(f(z))\| \leq \frac{4\gamma}{(\lambda-1)(1+n\gamma-2\gamma)}.$$

So we need to estimate $\|f(z)\|$. An estimate for this norm is not immediately forthcoming, but an extremely poor estimate presents itself if we consider that we can write $f(z)$ in the χ basis using the change of basis matrix presented in section 4.3.

Writing $f(z)$ in the form that we have it in the basis $\{1, z, z^{-1}, z^2, \ldots\}$ is not particularly easy, but note that

$$\|M_z - \lambda)(f(z))\| = \|(M_z - \lambda)(z^q f(z))\|$$

for any q since multiplication by z is unitary. Hence we can simply use a different function which is much more easily written in the preferred basis without destroying the information that we worked so hard to obtain. Depending on the parity of n, either the leading term or the constant term will become the coefficient of the last power of z in $\{1, z, z^{-1}, z^2, \ldots\}$. Hence if n is even, we will use the form of the approximate eiegenvector presented in Lemma 4.2 and if n is odd, we will use Lemma 4.3.

The important fact here is that the last row of the change of basis matrix only has one term in it, hence we know that

$$z^q f(z) = \|\Phi_{n-1}(z)\| \, \chi_{n-1} + \cdots$$

if we change basis to $\{\chi_0, \chi_1, \ldots, \chi_{n-1}\}$. Furthermore since this is an orthonormal basis, we know that

$$\|f(z)\| \geq \|\Phi_{n-1}(z)\| \, .$$

Thus if we can approximate this, we have a lower bound for the norm in which we are interested. So we have the following lemma:

Lemma 4.5. *If $\{\Phi_n\}_{n\geq 0}$ are the monic orthogonal polynomials associated with $\mu = (1-\gamma)\lambda + \gamma\delta_1$, then*

$$\|\Phi_n\| \geq 1 - \gamma, \quad \forall n \geq 0.$$

Proof. We know from 1.13 that

$$\begin{aligned}\|\Phi_n\| &= \prod_{k=0}^{n-1} \rho_k \\ &= \prod_{k=0}^{n-1} \sqrt{1 - |\alpha_k|^2}.\end{aligned}$$

So

$$\lim_{n\to\infty} \|\Phi_n\| = \prod_{k=0}^{\infty} \rho_k.$$

Hence we have

$$\begin{aligned}\log\left(\lim_{n\to\infty}\|\Phi_n\|\right) &= \sum_{k=0}^{\infty}\log\rho_k \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\log\left(1-|\alpha_k|^2\right).\end{aligned}$$

Since $|\alpha_k|^2<1$, we know that $\log\left(1-|\alpha_k|^2\right)=-\sum_{j=1}^{\infty}\frac{1}{j}|\alpha_k|^{2j}$. Hence

$$\begin{aligned}\log\left(\lim_{n\to\infty}\|\Phi_n\|\right) &= -\frac{1}{2}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}|\alpha_k|^{2j} \\ &= -\sum_{j=1}^{\infty}\frac{1}{2j}\sum_{k=0}^{\infty}|\alpha_k|^{2j}.\end{aligned}$$

Examining the inner sum, Theorem 2.2 and Abel's Summation Formula give us

$$\begin{aligned}\sum_{0\le k<x}|\alpha_k|^{2j} &= \sum_{0\le k<x}\left(\frac{1}{\gamma^{-1}+k}\right)^{2j} \\ &= \frac{\lfloor x+1\rfloor}{(\gamma^{-1}+t)^{2j}}+2j\int_0^x\frac{\lfloor t+1\rfloor}{(\gamma^{-1}+t)^{2j+1}}dt.\end{aligned}$$

Hence since $2j>1$ for all $j\ge 1$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}|\alpha_k|^{2j} &= 2j\int_0^{\infty}\frac{\lfloor t+1\rfloor}{(\gamma^{-1}+t)^{2j+1}}dt \\ &= 2j\int_{\gamma^{-1}}^{\infty}\frac{\lfloor u-\gamma^{-1}+1\rfloor}{u^{2j+1}}du \\ &\le 2j\int_{\gamma^{-1}}^{\infty}\frac{u-\gamma^{-1}+1}{u^{2j+1}}du \\ &\le 2j\int_{\gamma^{-1}}^{\infty}\frac{u+1}{u^{2j+1}}du \\ &= 2j\int_{\gamma^{-1}}^{\infty}\left(u^{-2j}+u^{-2j-1}\right)du \\ &= 2j\left(-\frac{1}{2j-1}u^{-2j+1}-\frac{1}{2j}u^{-2j}\right)\Bigg|_{\gamma^{-1}}^{\infty} \\ &= 2j\left(\frac{\gamma^{2j-1}}{2j-1}+\frac{\gamma^{2j}}{2j}\right).\end{aligned}$$

Hence we have

$$\log\left(\lim_{n\to\infty}\|\Phi_n\|\right) \geq -\sum_{j=1}^{\infty}\frac{1}{2j}\left(2j\left(\frac{\gamma^{2j-1}}{2j-1}+\frac{\gamma^{2j}}{2j}\right)\right)$$
$$= -\sum_{m=1}^{\infty}\frac{\gamma^m}{m}.$$

But since $|\gamma| < 1$, this gives us

$$\log\left(\lim_{n\to\infty}\|\Phi_n\|\right) \geq \log\left(1-\gamma\right).$$

Hence

$$\lim_{n\to\infty}\|\Phi_n\| \geq 1-\gamma.$$

But since $\frac{\|\Phi_n\|}{\|\Phi_{n-1}\|} = \rho_{n-1}$ and $\rho_{n-1} < 1$, this means that $\{\|\Phi_n\|\}_{n=0}^{\infty}$ is a strictly decreasing sequence of real numbers, hence

$$\|\Phi_n\| \geq 1-\gamma, \quad \forall n \geq 0$$

as desired. □

And with that, we get the following estimate:

$$\left\|(M_z-\lambda)\left(\frac{f(z)}{\|f(z)\|}\right)\right\| \leq \frac{4\gamma}{(1-\gamma)(\lambda-1)(1+n\gamma-2\gamma)}$$

Since $\frac{f(z)}{\|f(z)\|}$ is a unit vector, we therefore know that there is an eigenvalue λ' such that

$$|\lambda'-\lambda| \leq \frac{4\gamma}{(1-\gamma)(\lambda-1)(1+n\gamma-2\gamma)}$$

5 "Almost Free" CMV Matrices

5.1 A Review of Clock Theorems

We conjectured in Section 4 that most of the zeros of $\Phi_{n,\beta}$ for inserted mass point converge to roots of $\overline{\beta}$ as $n \to \infty$. In particular, this would mean that the distance between neighboring zeros would converge to $2\pi/n$. This last condition is called *clock behavior*. We will now state some known results concerning this behavior to help put our result into context.

Theorem 5.1 (Theorem 4.1 of [7])**.** *Let $\{\alpha_j\}_{j\geq 0}$ be a sequence of Verblunsky coefficients satisfying*

$$\sum_{j=0}^{\infty} |\alpha_j| < \infty \tag{5.1}$$

and let $\{\beta_j\}_{j\geq 1}$ be a sequence of points on $\partial\mathbb{D}$. Let $\{\theta_j^{(n)}\}_{j=1}^{n}$ be defined so $0 \leq \theta_1^{(n)} \leq \cdots \leq \theta_n^{(n)} < 2\pi$ and so that $e^{i\theta_j^{(n)}}$ are the zeros of the paraorthogonal polynomials Φ_{n,β_j}. Then (with $\theta_{n+1}^{(n)} \equiv \theta_1^{(n)+2\pi}$)

$$\sup_j\, n\left|\theta_{j+1}^{(n)} - \theta_j^{(n)} - \frac{2\pi}{n}\right| \to 0$$

as $n \to \infty$.

This result does not apply to inserted mass point; in that case there exists C such that $|\alpha_n| \geq C/n\ \forall n$ and hence

$$\sum_{n=0}^{\infty} |\alpha_n| \geq \sum_{n=0}^{\infty} \frac{C}{n} = \infty.$$

Theorem 5.2 (Theorem 1.7 in [1])**.** *If $\mathbb{E}\{|\alpha_k|^2\} = o(k^{-1})$, then (with probability 1) the spacing between eigenvalues converges to $2\pi/n$.*

Inserted mass point satisfies the hypothesis of this theorem since we can pick C such that $|\alpha_k| \leq C/k\ \forall k$ and get

$$\lim_{k\to\infty} \frac{|\alpha_k|^2}{k^{-1}} \leq \lim_{k\to\infty} \frac{C^2}{k} = 0.$$

This puts inserted mass point in an interesting position; while Theorem 5.2 tells us that we should expect clock behavior, this is not guaranteed by Theorem 5.1. Indeed, Simon conjectures in [7] that Theorem 5.1 stills holds if $|\alpha_n| = C_0 n^{-1} + \text{error}$ with either $\text{error} \in \ell^1$ or $|\text{error}| \leq Cn^{-2}$.

5.2 Shifted CMV Matrices

One interesting property of inserted mass point is that if we shift the Verblunsky coefficients we get Verblunsky coefficients for a different inserted mass point:

Lemma 5.3. *Let $\alpha_k(\omega,\gamma)=\alpha_k(\mu_{\omega,\gamma})$. Then for any n and m we have*

$$\alpha_{n+m}(\omega,\gamma)=\alpha_n(\omega^{-m},(1/\gamma+m)^{-1}). \tag{5.2}$$

Proof.

$$\alpha_{n+m}(\omega,\gamma)=\frac{\omega^{-(n+m+1)}}{1/\gamma+n+m}=\frac{(\omega^{-m})^{-(n+1)}}{(1/\gamma+m)+n}=\alpha_n(\omega^{-m},(1/\gamma+m)^{-1}).$$

□

Since

$$\lim_{m\to\infty}\frac{(1/\gamma+m)^{-1}}{\gamma}=\lim_{m\to\infty}\frac{1}{1+\gamma m}=0$$

we see that increasing m will make γ arbitrarily small, in some sense bringing us closer to Lebesgue measure. This motivates the following definitions:

Definition 5.4 (Shifted CMV Matrix)**.** *For a CMV matrix $\mathcal{C}=\mathcal{C}(\alpha_0,\alpha_1,\ldots)$ and $\beta\in\partial\mathbb{D}$ define the* shifted CMV matrix $\mathcal{C}_\beta^{(n,m)}$ *by*

$$\mathcal{C}_\beta^{(n,m)}=\mathcal{C}(\alpha_m,\alpha_{m+1},\ldots,\alpha_{n-2},\beta). \tag{5.3}$$

Definition 5.5. *Let $\mathcal{C}=\mathcal{C}(\alpha_0,\alpha_1,\ldots)$ be a CMV matrix. Define functions d and ℓ by*

$$d(n,m)=\max_i\ \min_j|\lambda_i-\mu_j|,\qquad\qquad \ell(n,m)=\min_{i\neq j}|\lambda_i-\lambda_j|.$$

where $\{\mu_i\}$ are the eigenvalues of $\mathcal{C}_\beta^{(n,m)}$ and $\{\lambda_i\}$ the eigenvalues of $\Lambda_\beta^{(n-m)}$.

Definition 5.6 (Almost Free)**.** *Let $\mathcal{C}=\mathcal{C}(\alpha_0,\alpha_1,\ldots)$ be a CMV matrix. If for all $\epsilon>0$ there exists N such that for all $n>N$ there is an m_n satisfying $d(n,m_n)<\epsilon\ell(n,m_n)$ and*

$$\lim_{n\to\infty}\frac{m_n}{n}=0$$

we say that C is almost free*.*

Lemma 5.7. *For $m<n$ and $n\geq 2$ we have*

$$\ell(n,m)\geq 2\sin(\tfrac{\pi}{n})\geq 4n^{-1}. \tag{5.4}$$

Proof. Note that, since the eigenvalues of $\Lambda_\beta^{(n-m)}$ are solutions of $z^{n-m} - \overline{\beta}$, the distance between any two neighboring eigenvalues will be

$$\begin{aligned}\ell(n,m) &= |e^{i\frac{2\pi}{n-m}} - 1| = \sqrt{2 - 2\cos\tfrac{2\pi}{n-m}} \\ &= \sqrt{4\sin^2\tfrac{\pi}{n-m}} = 2\sin(\tfrac{\pi}{n-m}) \\ &\geq 2\sin(\tfrac{\pi}{n}).\end{aligned}$$

Since $\sin\theta$ is concave down on $(0,\pi/2)$ we have have $\sin\theta \geq \frac{2}{\pi}\theta$ for any $\theta \in (0,\pi/2)$, so in particular if $n \geq 2$ we have $\sin(\pi/n) \geq (2/\pi)(\pi/n) = 2/n$. ☐

5.3 Norm Estimates

In this section we find sufficient conditions on Verblunsky coefficients for a CMV matrix to be almost free.

Definition 5.8. *For A an $n \times n$ matrix with ijth entry a_{ij}, define $\|A\|_2$ by*

$$\|A\|_2 = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\right)^{1/2} \tag{5.5}$$

Lemma 5.9. *For any $n \times n$ matrix A, $\|A\| \leq \|A\|_2$.*

Proof. Let a_{ij} be the ijth entry of A and let a_i be its ith row. If $\mathbf{x} = (x_0, x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ satisfies $\|\mathbf{x}\| = 1$ then

$$\|A\mathbf{x}\|^2 = \sum_{i=1}^n \left|\sum_{j=1}^n a_{ij}x_j\right|^2 = \sum_{i=1}^n |\langle a_i, \mathbf{x}\rangle|^2 \leq \sum_{i=1}^n \|a_i\|^2 = \|A\|_2^2.$$

☐

Lemma 5.10. *Let A be a normal $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\|A\| = \max_i |\lambda_i|$.*

Proof. Since A is normal there exists an orthonormal basis of eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $x = \sum_{i=1}^n \alpha_i \mathbf{e}_i$ satisfy $\|\mathbf{x}\| = 1$. Then

$$\begin{aligned}\|A\mathbf{x}\|^2 &= \left\|A\sum_{i=1}^n \alpha_i\mathbf{e}_i\right\|^2 = \left\|\sum_{i=1}^n \alpha_i\lambda_i\mathbf{e}_i\right\|^2 = \sum_{i=1}^n |\alpha_i|^2|\lambda_i|^2 \\ &\leq \sum_{i=1}^n |\alpha_i|^2 \left(\max_j |\lambda_j|\right)^2 = \left(\max_j |\lambda_j|\right)^2.\end{aligned}$$

To show the other inequality let λ_k be the eigenvalue of A with maximal absolute value. Then $A\mathbf{e}_k = \lambda_k \mathbf{e}_k \Longrightarrow \|A\mathbf{e}_k\| = |\lambda_k| \Longrightarrow \|A\| \geq |\lambda_k|$. $\square$

Lemma 5.11. *Let A be a normal $n \times n$ matrix and B an arbitrary $n \times n$ matrix. If $\|A - B\| < \epsilon$ for some $\epsilon > 0$, then for any eigenvalue λ of B there exists an eigenvalue μ of A such that $|\lambda - \mu| < \epsilon$.*

Proof. If λ is an eigenvalue of A then we're done, so assume that λ is not an eigenvalue of A, i.e. that $A - \lambda I$ is invertible. Since λ is an eigenvalue of B there must exist some $\mathbf{w} \in \mathbb{C}^n$ such that $B\mathbf{w} = \lambda\mathbf{w}$. This gives

$$\epsilon \|\mathbf{w}\| > \|(A - B)\mathbf{w}\| = \|A\mathbf{w} - \lambda\mathbf{w}\| = \|(A - \lambda I)\mathbf{w}\|.$$

Now we set $\mathbf{v} = (A - \lambda I)\mathbf{w}$ and observe that the above inequality becomes

$$\begin{aligned} \epsilon \left\|(A - \lambda I)^{-1}\mathbf{v}\right\| &> \left\|(A - \lambda I)(A - \lambda I)^{-1}\mathbf{v}\right\| \\ \left\|(A - \lambda)^{-1}\mathbf{v}\right\| &> \frac{1}{\epsilon}\|\mathbf{v}\| \end{aligned}$$

which implies $\left\|(A - \lambda)^{-1}\right\| > 1/\epsilon$, and hence by Lemma 5.10 that $(A - \lambda)^{-1}$ must have at least one eigenvalue $\widetilde{\mu}$ satisfying $|\widetilde{\mu}| > 1/\epsilon$. Thus $A - \lambda I$ has $\widetilde{\mu}^{-1}$ as an eigenvalue for some eigenvector $\mathbf{y}$ and we get

$$\begin{aligned} (A - \lambda I)\mathbf{y} &= \widetilde{\mu}^{-1}\mathbf{y} \\ A\mathbf{y} &= (\widetilde{\mu}^{-1} + \lambda)\mathbf{y}. \end{aligned}$$

Therefore $\mu = \widetilde{\mu}^{-1} + \lambda$ is an eigenvalue of A satisfying $|\mu - \lambda| = |\widetilde{\mu}^{-1}| < \epsilon$. $\square$

Theorem 5.12. *Let $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots)$ be a CMV matrix and fix $\beta \in \partial\mathbb{D}$. Then*

$$\left\|\mathcal{C}_\beta^{(n)} - \Lambda_\beta^{(n)}\right\|^2 \leq 4 \sum_{k=m}^{n-2} |\alpha_k|^2. \tag{5.6}$$

Remark. Simon [6] gives a much more general estimate for infinite CMV matrices using the $\mathcal{LM}$ factorization, which in our case reduces to

$$\left\|C_\beta^{(n)} - \Lambda_\beta^{(n)}\right\|^2 \leq 12 \sum_{k=0}^{\infty} \left(1 - \sqrt{1 - |\alpha_k|^2}\right).$$

Proof. We will first find $\|\mathcal{C}_\beta^{(n)} - \Lambda_\beta^{(n)}\|_2^2$ and then apply Lemma 5.9. To simplify computation, define the following sequences to the five diagonals of $\mathcal{C}_\beta^{(n)} - \Lambda_\beta^{(n)}$ omitting both the first terms of each diagonal and those involving β (d_0 is the main diagonal, d_1 the first diagonal above it, etc):

$$\begin{aligned}
d_0 &= -\overline{\alpha}_1\alpha_0, -\overline{\alpha}_2\alpha_1, -\overline{\alpha}_3\alpha_2, \ldots \\
d_1 &= -\rho_1\alpha_0, \overline{\alpha}_3\rho_2, -\rho_3\alpha_2, \overline{\alpha}_5\rho_4, -\rho_5\alpha_4, \ldots \\
d_{-1} &= \overline{\alpha}_2\rho_1, -\rho_2\alpha_1, \overline{\alpha}_4\rho_3, -\rho_4\alpha_3, \overline{\alpha}_6\rho_5, \ldots \\
d_2 &= 0, \rho_3\rho_2 - 1, 0, \rho_5\rho_4 - 1, 0, \rho_7\rho_6 - 1, \ldots \\
d_{-2} &= \rho_2\rho_1 - 1, 0, \rho_4\rho_3 - 1, 0, \rho_6\rho_5 - 1, 0, \ldots
\end{aligned}$$

From the main diagonal, we get

$$\sum_{k=1}^{n-2} |d_0(k)|^2 = \sum_{k=1}^{n-2} |-\overline{\alpha}_k\alpha_{k-1}|^2 = \sum_{k=1}^{n-2} |\alpha_k\alpha_{k-1}|^2.$$

Combining d_1 and d_{-1} we get two new sums

$$\sum_{k=1}^{n-2} |-\rho_k\alpha_{k-1}|^2 + \sum_{k=1}^{n-3} |\overline{\alpha}_{k+1}\rho_k|^2.$$

Since the $(n-2)$th term involves β, the second sum only goes up to $k = n-3$. The first sum yields

$$\begin{aligned}
\sum_{k=1}^{n-2} |-\rho_k\alpha_{k-2}|^2 &= \sum_{k=1}^{n-2} |\alpha_{k-1}|^2(1 - |\alpha_k|^2) = \sum_{k=1}^{n-2} \left[|\alpha_{k-1}|^2 - |\alpha_{k-2}|^2|\alpha_k|^2\right] \\
&= \sum_{k=1}^{n-2} |\alpha_{k-1}|^2 - \sum_{k=1}^{n-2} |\alpha_{k-1}\alpha_k|^2
\end{aligned}$$

and for the second we get

$$\begin{aligned}
\sum_{k=1}^{n-3} |\overline{\alpha}_{k+1}\rho_k|^2 &= \sum_{k=1}^{n-3} |\alpha_{k+1}|^2(1 - |\alpha_k|^2) = \sum_{k=1}^{n-3} \left[|\alpha_{k+1}|^2 - |\alpha_{k+1}|^2|\alpha_k|^2\right] \\
&= \sum_{k=1}^{n-3} |\alpha_{k+1}|^2 - \sum_{k=1}^{n-3} |\alpha_k\alpha_{k+1}|^2.
\end{aligned}$$

Combining d_2 and d_{-2} we get

$$\begin{aligned}\sum_{k=1}^{n-3}(1-\rho_k\rho_{k+1})^2 &= \sum_{k=1}^{n-3}(1-2\rho_k\rho_{k+1}+\rho_k^2\rho_{k+1}^2)\\ &= \sum_{k=1}^{n-3}\left[1-2\rho_k\rho_{k+1}+(1-|\alpha_k|^2)(1-|\alpha_{k+1}|^2)\right]\\ &= \sum_{k=1}^{n-3}\left[2(1-\rho_k\rho_{k+1})+|\alpha_k\alpha_{k+1}|^2-|\alpha_k|^2-|\alpha_{k+1}|^2\right]\end{aligned}$$

which gives us four sums:

$$2\sum_{k=1}^{n-3}(1-\rho_k\rho_{k+1})+\sum_{k=1}^{n-3}|\alpha_k\alpha_{k+1}|^2-\sum_{k=1}^{n-3}|\alpha_k|^2-\sum_{k=1}^{n-3}|\alpha_{k+1}|^2.$$

Since $xy \le x^2+y^2$ for $x, y \ge 0$, we can get a simple bound for the first term:

$$\begin{aligned}2\sum_{k=1}^{n-3}(1-\rho_k\rho_{k+1}) &\le 2\sum_{k=1}^{n-3}\left[1-\rho_k^2+1-\rho_{k+1}^2\right]\\ &= 2\sum_{k=1}^{n-3}|\alpha_k|^2+2\sum_{k=1}^{n-3}|\alpha_{k+1}|^2.\end{aligned}$$

The first entries of the diagonals contribute

$$\begin{aligned}|\alpha_0|^2+|\rho_0\alpha_1|^2+|\rho_0-1|^2 &\le |\alpha_0|^2+(1-|\alpha_0|^2)|\alpha_1|^2+|\rho_0^2-1|^2\\ &= |\alpha_0|^2+|\alpha_1|^2-|\alpha_0\alpha_1|^2-|\alpha_0|^4\end{aligned}$$

and the terms with β will give

$$\begin{aligned}|\beta\rho_{n-2}-\beta|^2+|\beta\alpha_{n-2}|^2 &= (\rho_{n-2}-1)^2+|\alpha_{n-2}|^2\\ &\le (\rho_{n-2}^2-1)^2+|\alpha_{n-2}|^2\\ &= |\alpha_{n-2}|^4+|\alpha_{n-2}|^2.\end{aligned}$$

Note that all of the sums involving $|\alpha_k\alpha_{k+1}|^2$ cancel. Reindexing sums involving $|\alpha_k|^2$ we see that they add up to

$$4\sum_{k=2}^{n-3}|\alpha_k|^2+|\alpha_0|^2+2|\alpha_1|^2+2|\alpha_{n-2}|^2.$$

Adding the first entries of the diagonals and those involving β we get

$$4\sum_{k=2}^{n-3}|\alpha_k|^2+2|\alpha_0|^2+3|\alpha_1|^2+3|\alpha_{n-2}|^2+|\alpha_{n-2}|^4-|\alpha_0\alpha_1|^2-|\alpha_0|^4$$

which is less than or equal to (throwing out negative terms and replacing $|\alpha_{n-1}|^4$ with $|\alpha_{n-1}|^2$)

$$4\sum_{k=2}^{n-3}|\alpha_k|^2+2|\alpha_0|^2+3|\alpha_1|^2+4|\alpha_{n-2}|^2\leq 4\sum_{k=0}^{n-2}|\alpha_k|^2.$$

Thus by Lemma 5.9 we have

$$\|\mathcal{C}_\beta^{(n)}-\Lambda_\beta^{(n)}\|^2\leq 4\sum_{k=0}^{n-2}|\alpha_k|^2$$

as desired. □

Corollary 5.13. *For every eigenvalue λ of $\Lambda_\beta^{(n-m)}$ there exists an eigenvalue μ of $\mathcal{C}_\beta^{(n,m)}$ such that*

$$|\mu-\lambda|^2\leq 4\sum_{k=m}^{n-2}|\alpha_k|^2. \tag{5.7}$$

Equivalently we have

$$d(n,m)^2\leq 4\sum_{k=m}^{n-2}|\alpha_k|^2. \tag{5.8}$$

Proof. This follows directly from Theorem 5.12 and Lemma 5.11. □

5.4 Necessary Conditions

Example 5.14. We will show that the CMV matrix for the Rogers-Szegő polynomials is almost free. For some $q\in(0,1)$ the Rogers-Szegő polynomials are the orthogonal polynomials for the measure

$$d\mu(\theta)=(2\pi\log(1/q))^{-1/2}\sum_{j=-\infty}^{\infty}q^{-(\theta-2\pi j)^2/2}d\theta$$

with Verblunsky coefficients $\alpha_k = (-1)^k q^{(k+1)/2}$.

Then

$$\sum_{k=m}^{n-1} |\alpha_k|^2 = \sum_{k=m}^{n-1} q^{m+1} = \frac{q^{m+1} - q^{n+1}}{1-q} \leq \frac{q^m}{1-q} \tag{5.9}$$

which, by Corollary 5.13, means that $d(n,m)^2 \leq 4q^m/(1-q)$. With constants $r \in (0,1)$ and K where

$$r = \sqrt{q}, \qquad K = \sqrt{4/(1-q)} \tag{5.10}$$

this relation becomes

$$d(n,m) \leq Kr^m. \tag{5.11}$$

Given $\epsilon > 0$ we want to find the smallest possible m satisfying $d(n,m) < \epsilon\ell(n,m)$. Using Lemma 5.7 we see that

$$Kr^m < \frac{4\epsilon}{n} \tag{5.12}$$

is sufficient. Since both sides of this inequality are strictly positive, we can take their logarithms to get

$$m \log r + \log K < \log(4\epsilon/n) \tag{5.13}$$

$$m > \frac{\log((4\epsilon/K)n^{-1})}{\log r} \tag{5.14}$$

$$= \frac{\log[K/(4\epsilon)] + \log n}{\log(1/r)} \tag{5.15}$$

where all of the terms except perhaps $\log[K/(2\epsilon)]$ are now positive. Thus

$$m = m_n = \left\lceil \frac{\log[K/(4\epsilon)] + \log n}{\log(1/r)} \right\rceil \tag{5.16}$$

guarantees (5.12). Clearly

$$\lim_{n\to\infty} \frac{m_n}{n} = \lim_{n\to\infty} \frac{1}{n} \left\lceil \frac{\log[K/(4\epsilon)] + \log n}{\log(1/r)} \right\rceil = 0. \tag{5.17}$$

This also implies that certainly $m_n < n - 1$ for all sufficiently large n.

Now we use Corollary 5.13 to show to that a much broader category of CMV matrices are almost free:

Theorem 5.15. *Let $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots)$ be a CMV matrix. If*

$$\sum_{k=m}^{\infty} |\alpha_k|^2 \leq \frac{M}{m^\gamma} \tag{5.18}$$

for some $\gamma > 2$ and $M > 0$, then $\mathcal{C}$ is almost free.

Proof. Fix $\epsilon > 0$. If we let $K = 2\sqrt{M}$ and $\beta = \gamma/2$ then by Corollary 5.13 we have $d(n,m) \leq K/m^\beta$. By Lemma 5.7 we need only can find $m < n-1$ such that

$$\frac{K}{m^\beta} < \frac{4\epsilon}{n} \tag{5.19}$$

to guarantee $d(n,m) < \epsilon(\ell(n,m))$. Solving for m, this inequality becomes

$$m > \left(\frac{Kn}{4\epsilon}\right)^{1/\beta}. \tag{5.20}$$

Thus choosing

$$m_n = \left\lceil \left(\frac{Kn}{4\epsilon}\right)^{1/\beta} \right\rceil \tag{5.21}$$

will satisfy (5.19). Of course we may not have $m_n < n-1$ for some n. However,

$$\begin{aligned}
\lim_{n\to\infty} \frac{m_n}{n} &= \lim_{n\to\infty} \frac{1}{n} \left\lceil \left(\frac{Kn}{4\epsilon}\right)^{1/\beta} \right\rceil && (5.22)\\
&\leq \lim_{n\to\infty} \frac{1}{n} \left[\left(\frac{K}{4\epsilon}\right)^{1/\beta} n^{1/\beta} + 1 \right] && (5.23)\\
&= \left(\frac{K}{4\epsilon}\right)^{1/\beta} \lim_{n\to\infty} n^{1/\beta - 1} + \lim_{n\to\infty} \frac{1}{n} && (5.24)\\
&= 0 && (5.25)
\end{aligned}$$

(since $1/\beta - 1 < 0$); thus we must have $m_n < n-1$ for all n sufficiently large. □

Theorem 5.16. *Let $d\mu = w(\theta)\frac{d\theta}{2\pi}$ with weight function $w(\theta) > 0$, $\inf_\theta w(\theta) = \alpha > 0$, and $\sum |c_i(d\mu)| < \infty$. Then*

$$\sup\left\{ \beta \Big| \sum_{n=1}^{\infty} (1+|n|)^\beta |c_n(d\mu)| < \infty \right\} = \sup\left\{ \beta \Big| \sum_{n=1}^{\infty} (1+|n|)^\beta |\alpha_n(d\mu)| < \infty \right\}$$

Remark. This is stated in Simon [4] as an immediate consequence of the extended Baxter's theorem, and will not be proved here.

Theorem 5.17. *Assume that there exist $p > 1$ and $M > 0$ such that, for all $n > 0$, $|c_n| < M/n^p$. Then for all $\epsilon \in (0,1)$ there exists K such that*

$$\sum_{n=m}^{\infty} |\alpha_n|^2 \leq \frac{K}{m^{2(p-1)-\epsilon}}. \tag{5.26}$$

Proof. We know from Theorem 5.16 that

$$c = \sup\left\{\beta \Big| \sum_{n=0}^{\infty}(1+n)^{\beta}|\alpha_n| < \infty\right\} = \sup\left\{\beta \Big| \sum_{n=0}^{\infty}(1+n)^{\beta}|c_n| < \infty\right\}, \tag{5.27}$$

and since $|c_n| \leq M/n^p \ \forall n > 0$ we have

$$\sum_{n=0}^{\infty}(1+n)^{\beta}|c_n| \leq |c_0| + M\sum_{n=1}^{\infty}\frac{(1+n)^{\beta}}{n^p} \tag{5.28}$$

$$\leq 1 + M\sum_{n=1}^{\infty}\frac{(2n)^{\beta}}{n^p} \tag{5.29}$$

$$= 1 + 2^{\beta}M\sum_{n=1}^{\infty}\frac{1}{n^{p-\beta}} \tag{5.30}$$

which converges for $p - \beta > 1 \implies \beta < p - 1$. Therefore $c \geq p - 1$ and we have that for all $\epsilon_1 \in (0,1)$

$$\sum_{n=0}^{\infty}(1+n)^{p-1-\epsilon_1}|\alpha_n| < \infty. \tag{5.31}$$

For convenience let $q = p - 1 - \epsilon_1$. Now for any $m \in \mathbb{N}$ we have

$$\sum_{n=0}^{\infty}(1+n)^q|\alpha_n| = \sum_{n=0}^{m-1}(1+n)^q|\alpha_n| + \sum_{n=m}^{\infty}(1+n)^q|\alpha_n|$$

and hence

$$\begin{aligned}
\left(\sum_{n=0}^{\infty}(1+n)^q|\alpha_n|\right)^2 &= \left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n| + \sum_{n=m}^{\infty}(1+n)^q|\alpha_n|\right)^2 \\
&\geq \left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n|\right)^2 + \left(\sum_{n=m}^{\infty}(1+n)^q|\alpha_n|\right)^2 \\
&\geq \left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n|\right)^2 + \sum_{n=m}^{\infty}(1+n)^{2q}|\alpha_n|^2 \\
&\geq \left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n|\right)^2 + m^{2q}\sum_{n=m}^{\infty}|\alpha_n|^2
\end{aligned}$$

For any $\epsilon_2 > 0$ we therefore have

$$m^{-\epsilon_2}\left(\sum_{n=0}^{\infty}(1+n)^q|\alpha_n|\right)^2 \geq m^{-\epsilon_2}\left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n|\right)^2 + m^{2(q-\epsilon_2)}\sum_{n=m}^{\infty}|\alpha_n|^2$$

which since

$$\lim_{m\to\infty} m^{-\epsilon_2}\left(\sum_{n=0}^{\infty}(1+n)^q|\alpha_n|\right)^2 = 0$$

means

$$\lim_{m\to\infty}\left[m^{-\epsilon_2}\left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n|\right)^2 + m^{2(q-\epsilon_2)}\sum_{n=m}^{\infty}|\alpha_n|^2\right] = 0.$$

By (5.31) we can rewrite this as

$$\begin{aligned}
0 &= \left(\lim_{m\to\infty} m^{-\epsilon_2}\right)\lim_{m\to\infty}\left(\sum_{n=0}^{m-1}(1+n)^q|\alpha_n|\right)^2 + \lim_{m\to\infty} m^{2(q-\epsilon_2)}\sum_{n=m}^{\infty}|\alpha_n|^2 \\
&= \lim_{m\to\infty} m^{2(q-\epsilon_2)}\sum_{n=m}^{\infty}|\alpha_n|^2.
\end{aligned}$$

Given $\epsilon > 0$ pick $\epsilon_1 = \epsilon_2 = \epsilon/4$. Then

$$2(q-\epsilon_2) = 2(p-1-\epsilon/4-\epsilon/4) = 2(p-1)-\epsilon$$

and hence

$$\lim_{m\to\infty} m^{2(p-1)-\epsilon} \sum_{n=m}^{\infty} |\alpha_n|^2 = 0.$$

This tells us that there exists N such that for all $m > N$

$$m^{2(p-1)-\epsilon} \sum_{n=m}^{\infty} |\alpha_n|^2 < 1$$

or equivalently

$$\sum_{n=m}^{\infty} |\alpha_n|^2 < \frac{1}{m^{2(p-1)-\epsilon}}.$$

Let

$$S = \max_{m\leq N} \left(\sum_{n=m}^{\infty} |\alpha_n|^2 \right)$$

and choose $\widetilde{K}$ such that $\widetilde{K}/N^{2(p-1)-\epsilon} > S$. Then we have that, for all $m \leq N$,

$$\sum_{n=m}^{\infty} |\alpha_n|^2 < \frac{\widetilde{K}}{m^{2(p-1)-\epsilon}}.$$

With $K = \max(\widetilde{K}, 1)$ we have that for all m

$$\sum_{n=m}^{\infty} |\alpha_n|^2 < \frac{K}{m^{2(p-1)-\epsilon}}$$

as desired. □

Remark. We would like to thank Tim Heath for helping us with the proof of Theorem 5.17.

Corollary 5.18. *Let $\mathcal{C}$ be a CMV matrix whose moments satisfy the hypothesis of Theorem 5.17 for $p > 2$. Then $\mathcal{C}$ is almost free.*

Proof. By Theorem 5.17 we know that for any $\epsilon \in (0, 1)$ there exists K such that

$$\sum_{n=m}^{\infty} |\alpha_n|^2 < \frac{K}{m^{2(p-1)-\epsilon}}. \tag{5.32}$$

In particular we can choose ϵ such that $2(p-1) - \epsilon > 2$. Then by Theorem 5.15 $\mathcal{C}$ is almost free. □

Lemma 5.19. *Let $f\colon [0,2\pi] \to (0,\infty)$ be an r times differentiable function with bounded rth derivative, say $|f^r(\theta)| < M$ for all θ, and $f^k(0) = f^k(2\pi)$ for $k < r-1$. Then there exists K such that for all $n > 0$*

$$\left| \int_0^{2\pi} e^{-in\theta} f(\theta)\, dt \right| \leq \frac{K}{n^r}. \tag{5.33}$$

Proof. If $r = 0$ then we have

$$\left| \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta \right| \leq \int_0^{2\pi} |e^{-in\theta} f(\theta)|\, d\theta \leq 2\pi M = \frac{2\pi M}{n^0}.$$

Assume then that Lemma 5.19 holds for $r = k$ and let f satisfy the hypothesis of Lemma 5.19 for $r = k+1$. Using integration by parts we get

$$\begin{aligned} \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta &= \left.\frac{e^{-in\theta} f(\theta)}{-in}\right|_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} e^{-in\theta} f'(\theta)\, d\theta \\ &= \frac{1}{in} \int_0^{2\pi} e^{-in\theta} f'(\theta)\, d\theta. \end{aligned}$$

Now f' satisfies the hypothesis of Lemma 5.19 for $r = k$, so there exists K such that

$$\left| \int_0^{2\pi} e^{-in\theta} f'(\theta)\, dt \right| \leq \frac{K}{n^k}.$$

This gives

$$\left| \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta \right| = \frac{1}{n} \left| \int_0^{2\pi} e^{-in\theta} f'(\theta)\, dt \right| \leq \frac{1}{n} \left(\frac{K}{n^k} \right) = \frac{K}{n^{k+1}}.$$

□

Theorem 5.20. *Let $f : [0,2\pi] \to (0,\infty)$ satisfy the hypothesis of Lemma 5.19 for some $r > 2$. If $\int_0^{2\pi} f(\theta) d\theta = 2\pi$ then $d\mu(\theta) = f(\theta) \frac{d\theta}{2\pi}$ is a nontrivial probability measure whose CMV matrix is almost free.*

Proof. That $d\mu$ is a nontrivial probability measure is clear from the conditions on f. Using the K given by Lemma 5.19 we get

$$|c_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta \right| \leq \frac{K/2\pi}{n^r}$$

and hence by Corollary 5.18 that the CMV matrix for $d\mu$ is almost free. □

Remark. It is important to note that Theorem 5.1 more or less implies Theorem 5.15: Let $\gamma > 2$. If

$$\sum_{k=m}^{\infty} |\alpha_k|^2 \leq \frac{M}{m^\gamma}$$

then we must also have $|\alpha_m| \leq M/m^\gamma$. This gives

$$\sum_{k=0}^{\infty} |\alpha_k| = \sum_{k=0}^{\infty} \frac{M}{k^\gamma} < \infty$$

and hence by Theorem 5.1 that the distance between neighboring eigenvalues of $\mathcal{C}_\beta^{(n)} = \mathcal{C}_\beta^{(n,0)}$ approaches $2\pi/n$ and hence that they will converge to the eigenvalues of $\Lambda_{\beta'}^{(n)}$ for some $\beta' \in \partial\mathbb{D}$. The only technicality is that we may not have $\beta' = \beta$.

5.5 Properties of Almost Free CMV Matrices

In this section we'll investigate the properties of almost free CMV matrices, in particular how the eigenvalues of their finite CMV matrices compare to the those of the free CMV matrix of the same dimension. In order to do this, we'll need several facts from [4].

Lemma 5.21. *Let $\mathcal{C} = \mathcal{C}(\alpha_0, \alpha_1, \ldots)$ be a matrix. Then for all n and $m < n$ there exists a rank one matrix Q and a unitary $m \times m$ matrix K such that*

$$\mathcal{C}_\beta^{(n)} - Q = K \oplus B \tag{5.34}$$

where $B = \mathcal{C}_\beta^{(n,m)}$ if m is even and $(\mathcal{C}_\beta^{(n,m)})^\top$ if m is odd.

Remark. This is a less general version of Theorem 4.5.2 of [4].

Proof. Just as with full CMV matrices, we can write $\mathcal{C} = \mathcal{LM}$ with $\mathcal{L}$ and $\mathcal{M}$ defined as in (1.30) and (1.31). If m is even, let $\mathcal{M}'$ be $\mathcal{M}$ with Θ_{m-1} replaced by $\left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right)$ with $\beta = (1+\overline{\alpha}_{m-1})/(1+\alpha_m)$. It's straightforward to check that $\mathcal{M}-\mathcal{M}'$ has rank 1 and hence that $\mathcal{L}(\mathcal{M}-\mathcal{M}')$ has rank at most 1. Setting $Q = \mathcal{L}(\mathcal{M}-\mathcal{M}')$ we see that $\mathcal{C} - Q = \mathcal{LM}'$ has the desired form. A similar argument involving modifying $\mathcal{L}$ works for n odd. □

Lemma 5.22. *Let U and V be two $n \times n$ unitary matrices such that $U - V$ is rank 1 and denote the number of eigenvalues of the matrix A contained* *in the set D by $E_D(A)$. If $I = \{e^{i\theta} : |\theta - \theta_0| < \delta\}$ for some $\theta_0 \in [0, 2\pi]$ and $\delta < \pi$ then*

$$|E_I(U) - E_I(V)| \leq 1. \tag{5.35}$$

Remark. This is an immediate consequence of Equation 1.4.25 of [4].

Theorem 5.23. *Let $\mathcal{C}$ be an almost free CMV matrix and fix $\beta \in \partial\mathbb{D}$. For $\epsilon > 0$ pick n and m such that $d(n,m) < \epsilon\ell(n,m)$. Let $\{\lambda_i\}$ be the eigenvalues of $\Lambda_\beta^{(n-m)}$. Pick $\{\theta_1, \ldots, \theta_k\}$ such that $\lambda_k = e^{i\theta_k}$. Then for all but at most m intervals of the form*

$$I_k = \left\{e^{i\theta} : \theta \in (\theta_k - \epsilon\tfrac{2\pi}{n-m}, \theta_{k+1} + \epsilon\tfrac{2\pi}{n-m})\right\}$$

contain an eigenvalue of $\mathcal{C}_\beta^{(n)}$.

Proof. First we'll show that $E_{I_k}(\mathcal{C}_\beta^{(n,m)}) \geq 2$ for all k. Define

$$\phi_d(n,m) = \max_i \min_j |\lambda_i - \mu_j| \tag{5.36}$$

$$\phi_\ell(n,m) = \min_{i \neq j} |\lambda_i - \lambda_j| = \tfrac{2\pi}{n-m}. \tag{5.37}$$

Then $d(n,m) < \epsilon\ell(n,m)$ implies $\phi_d(n,m) < \ell\phi_\ell(n,m)$, so for each eigenvalue $e^{i\theta_k}$ of $\Lambda_\beta^{(m)}$ there exists an eigenvalue $e^{i\phi_k}$ of $\mathcal{C}_\beta^{(n,m)}$ such that $|\phi_k - \theta_k| < \epsilon\frac{2\pi}{n-m}$. This gives $e^{i\phi_k}, e^{i\phi_{k+1}} \in I_k$ and hence $E_{I_k}(\mathcal{C}_\beta^{(n,m)}) \geq 2$ as desired.

Lemmas 5.21 and 5.22 give

$$\left|E_{I_k}(\mathcal{C}_\beta^{(n)}) - E_{I_k}(K \oplus B)\right| \leq 1. \tag{5.38}$$

Since $E_{I_k}(K \oplus \mathcal{C}_\beta^{(n,m)}) = E_{I_k}(K) + E_{I_k}(B)$ this becomes

$$\left|E_{I_k}(\mathcal{C}_\beta^{(n)}) - E_{I_k}(K) - E_{I_k}(B)\right| \leq 1. \tag{5.39}$$

Since K is an $m \times m$ matrix it has at most m eigenvalues. Hence for all but at most m of the I_k we'll have $E_{I_k}(K) = 0$ and therefore

$$\left|E_{I_k}(\mathcal{C}_\beta^{(n)}) - E_{I_k}(B)\right| \leq 1. \tag{5.40}$$

Since $E_{I_k}(B) = E_{I_k}(\mathcal{C}_\beta^{(n,m)})$ (because the eigenvalues of a matrix A and its transpose are the same) we have $E_{I_k}(B) \geq 2$ and hence

$$E_{I_k}(\mathcal{C}_\beta^{(n)}) \geq 1. \tag{5.41}$$

□

Remark. Simon proves a very similar theorem about the interlacing of the zeros of paraorthogonal polynomials in [8].

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