POISSON STATISTICS FOR EIGENVALUES: FROM RANDOM SCHRÖDINGER OPERATORS TO RANDOM CMV MATRICES

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Abstract. We give a brief overview of the results and methods used in the study of the statistical distribution of eigenvalues for random Schrödinger operators and random CMV matrices.

1. Introduction

The study of the spectra of random Schrödinger operators and the distribution of their eigenvalues was initiated by the very important paper of Anderson [4]. Anderson showed that certain random lattices exhibit absence of diffusion, a phenomenon which is now called Anderson localization.

The first rigorous mathematical proof of the Anderson localization was obtained by Goldsheid, Molchanov and Pastur [10]. They studied a special case of the continuous one-dimensional random Schrödinger operator, where the potential was defined using Brownian motion on a Riemannian manifold.

A few years later, Fröhlich and Spencer analysed a model of discrete Schrödinger operator called the Anderson tight-binding model. In [8], they obtained the first proof of the Anderson localization for multi-dimensional Schrödinger operators. Different proofs of the Anderson localization were discovered later; we will only mention the the proof of Aizenman and Molchanov [3], which is relevant for the topic considered here.

Recent developments in the theory of orthogonal polynomials on the unit circle have emphasized the importance of a new class of unitary matrices, the CMV matrices. These matrices can be seen as unitary analogues of the self-adjoint Jacobi matrices (see [19] and [20] for a detailed presentation of the theory of orthogonal polynomials on the unit circle and CMV matrices). Many of the techniques used in the study

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of random Schrödinger operators can be adapted to random CMV matrices. Localization results for random orthogonal polynomials on the unit circle and random CMV matrices can be found in [23] and [20].

In addition to the Anderson localization, one can also study the statistical distribution of the eigenvalues. It turns out that in the regimes where the Anderson localization is expected, the local statistical distribution of the eigenvalues of the random Schrödinger operators is Poisson. This means that there is no local correlation between these eigenvalues.

The first result on the local structure of the spectrum of the one-dimensional Schrödinger operators was obtained by Molchanov in [16]. This result was extended to the multi-dimensional Anderson tight-binding model by Minami [14], using [3].

A similar result holds for the eigenvalues of random CMV matrices; see [21] and [22]. More precisely, for some classes of random CMV matrices, the local statistical distribution of the eigenvalues converges to the Poisson distribution. This is equivalent to the fact that the local distribution of the zeros of random paraorthogonal polynomials on the unit circle converges to the Poisson distribution. The result was extended from paraorthogonal polynomials to regular orthogonal polynomials by Davies and Simon in [7].

2. Random Schrödinger Operators

The one-dimensional continuum random Schrödinger operator studied by Goldsheid, Molchanov and Pastur was defined by:

$$H(\omega) = -\frac{d^2}{dt^2} + q(t,\omega), \qquad t \in \mathbb{R}, \quad \omega \in \Omega \tag{2.1}$$

where $q(t,\omega)$ is a stationary random potential.

More precisely,

$$q(t,\omega) = F(x_t(\omega)) \tag{2.2}$$

where $x_t(\omega)$ is the Brownian motion on a compact manifold K and $F : K \to \mathbb{R}$ be a smooth nonflat Morse function.

Goldsheid, Molchanov and Pastur proved in [10] that the operator $H = H(\omega)$ defined by (2.1) has pure point spectrum for almost all $\omega \in \Omega$ (i.e., a complete system of eigenvalues in $L^2(\mathbb{R})$). This result was strengthened in [15] by Molchanov, who showed that with probability 1, each eigenfunction of the operator $H = H(\omega)$ defined by (2.1) decreases exponentially. Thus, (2.1) was the first model of one-dimensional random Schrödinger operators for which a mathematical proof of the Anderson localization was obtained.

In addition to the Anderson localization, one can also study the local structure of the spectrum. The main question is whether the energy levels are correlated or not. This issue was settled by Molchanov, who proved in [16] that there is no local correlation between the eigenvalues of the operator defined by (2.1).

In order to describe Molchanov's result, let $H_V = H_V(\omega)$ be the random one-dimensional Schrödinger operator on $L^2(-V, V)$ with Dirichlet boundary conditions. Denote by $N_V(I)$ the number of eigenvalues of the operator H_V situated in the interval I. Then, for any fixed $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$ and any nonnegative integers $k_1, k_2, \ldots, k_n$,

$$\lim_{V\to\infty} \mathbb{P}\Big(N_V\big(E_0 + \frac{a_1}{2V}, E_0 + \frac{b_1}{2V}\big) = k_1, \ \ldots, \tag{2.3}$$

$$N_V\big(E_0 + \frac{a_n}{2V}, E_0 + \frac{b_n}{2V}\big) = k_n \Big) \tag{2.4}$$

$$= e^{-(b_1-a_1)\,n(E_0)}\ \frac{((b_1-a_1)\,n(E_0))^{k_1}}{k_1!}\ \cdots \tag{2.5}$$

$$e^{-(b_n-a_n)\,n(E_0)}\ \frac{((b_n-a_n)\,n(E_0))^{k_n}}{k_n!} \tag{2.6}$$

This means that the local statistical distribution of the eigenvalues of the operator H_V (rescaled near E_0) converges, as $V \to \infty$, to the Poisson point process with intensity measure $n(E_0)\,dx$, where dx denotes the Lebesgue measure. Hence, Molchanov's result [16] shows that there is no local correlation between the energy levels of the random one-dimensional operator (2.1).

The next challenge for mathematical physicists was to see whether a similar result holds for multi-dimensional Schrödinger operators. A positive answer to this question was given by Minami in [14], who showed that near energies where Anderson localization is expected (more precisely, in regimes where the Anderson localization holds via [3]), there is no correlation between the eigenvalues of the Anderson tight-binding model.

In order to describe Minami's result, let $H = -\Delta + V_\omega$ be the standard Anderson tight-binding model acting in $l^2(\mathbb{Z}^d)$ and let $H^\Lambda = \chi_\Lambda\, H\, \chi_\Lambda$ be the truncated Schrödinger operator corresponding to the hypercube $\Lambda \subset \mathbb{Z}^d$. Also let $G^\Lambda(z) = (H^\Lambda - z)^{-1}$ be the resolvent associated to H^Λ. If $E_1(\Lambda) \le E_2(\Lambda) \le \cdots \le E_n(\Lambda)$ are the eigenvalues of H^Λ, then we can define the integrated density of states:

$$N(E) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda|}\, \#\{j,\, E_j(\Lambda) \le E\} \tag{2.7}$$

and the density of states:

$$n(E) = dN(E)/dE \tag{2.8}$$

Note that if $n(E) > 0$, then the average spacing of the eigenvalues near the energy E is of order $|\Lambda|^{-1}$. Consider the rescaled spectrum (near E):

$$\xi_j(\Lambda, E) = |\Lambda| \, (E_j(\Lambda) - E) \tag{2.9}$$

and the corresponding point process

$$\xi(\Lambda, E) = \sum_j \delta_{\xi_j(\Lambda, E)} \tag{2.10}$$

Minami proved in [14] that, under suitable conditions:

$$\xi(\Lambda, E) \xrightarrow{\Lambda \uparrow \mathbb{Z}^d} \text{Poisson point process of intensity } n(E) \tag{2.11}$$

This means exactly that there is no correlation between the eigenvalues of H^Λ if Λ is large.

We should mention here that one of the main ingredients of the proof is the following technical lemma:

$$\mathbb{E}\left[\det \begin{pmatrix} \operatorname{Im} G^\Lambda(z; x, x) & \operatorname{Im} G^\Lambda(z; x, y) \\ \operatorname{Im} G^\Lambda(z; y, x) & \operatorname{Im} G^\Lambda(z; y, y) \end{pmatrix}\right] \leq C \tag{2.12}$$

where the constant C depends only on the distribution of the potential $V = V_\omega$. This result is referred to as the "Minami trick". It is worth to mention that (2.12) encodes a spectacular and somewhat mysterious cancelation that has found applications to other questions in mathematical physics; see for example [13].

3. Random CMV Matrices

Recent work of Cantero, Moral and Velázquez [5] emphasized the importance of a new class of unitary random matrices called now CMV matrices. These matrices are intimately connected with the orthogonal polynomials on the unit circle (see the monographs [19] and [20] on the theory of orthogonal polynomials on the unit circle).

The CMV matrix is a five-diagonal matrix realization for the unitary operator $z \to zf(z)$ on $L^2(\mathbb{T}; \mu)$, where μ is a non-trivial probability measure on the unit circle $\mathbb{T}$ (we call a measure non-trivial if it is not supported on finitely many points). For any such measure μ we can apply the Gram-Schmidt procedure to the set of polynomials $\{1, z, z^2, \ldots\} \in L^2(\mathbb{T}, \mu)$ and get the set of monic orthogonal polynomials $\{\Phi_0(z, d\mu), \Phi_1(z, d\mu), \Phi_2(z, d\mu), \ldots\} \in L^2(\mathbb{T}; \mu)$.

These polynomials obey the recurrence relation

$$\Phi_{k+1}(z, d\mu) = z\Phi_k(z, d\mu) - \overline{\alpha}_k \Phi_k^*(z, d\mu) \qquad k \geq 0 \tag{3.1}$$

where, for $\Phi_k(z) = \sum_{j=0}^{k} b_j z^j$, the reversed polynomial $\Phi_k^*(z)$ is defined by $\Phi_k^*(z) = \sum_{j=0}^{k} \bar{b}_{k-j} z^j$. The recurrence coefficients $\{\alpha_n\}_{n\geq 0}$ are called Verblunsky coefficents and are complex numbers of absolute value < 1.

If we apply the Gram-Schmidt algorithm to the sequence $\{ 1, z, z^{-1}, z^2, z^{-2} \dots \}$ we get the set $\{ \chi_0(z), \chi_1(z), \chi_2(z), \dots \}$, which is a basis of $L^2(\mathbb{T};\mu)$. The CMV matrix associated to the measure μ is the matrix representation of the operator $f(z) \to zf(z)$ on $L^2(\mathbb{T};\mu)$. It has the form:

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{3.2}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$

Note that the Jacobi matrices obtained in a similar way for orthogonal polynomials on the real line are tri-diagonal matrices. As in the case of orthogonal polynomials on the real line, an important connection between CMV matrices and monic orthogonal polynomials is

$$\Phi_n(z) = \det(zI - \mathcal{C}^{(n)}) \tag{3.3}$$

where $\mathcal{C}^{(n)}$ is the upper left $n \times n$ corner of $\mathcal{C}$.

If $|\alpha_{n-1}| = 1$, then the CMV matrix decouples between $(n-1)$ and n. The upper left corner is an $(n \times n)$ unitary matrix

$$\mathcal{C}^{(n)} = \mathcal{C}^{(n)}_{\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}} \tag{3.4}$$

We will consider random CMV matrices and study the statistical distribution of their eigenvalues. We will randomize the matrix $\mathcal{C}^{(n)}_{\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}}$ by taking independent identically distributed random variables α_0, α_1, …, α_{n-2}. The last variable, α_{n-1} will be chosen to be uniformly distributed on the unit circle. As in the case of random Schrödinger operators, for various classes of $n \times n$ random CMV matrices, the local statistical distribution of the eigenvalues of these matrices will converge (as $n \to \infty$) to the Poisson distribution. This fact indicates that, as n gets large, there is no local correlation between these eigenvalues. The following theorem can be found in [21] (and will be extended to singular distributions in Theorem 3.2):

Theorem 3.1. *Consider the random CMV matrices* $\mathcal{C}^{(n)} = \mathcal{C}^{(n)}_{\{\alpha_0,\alpha_1,\dots,\alpha_{n-1}\}}$ *where* $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$ *are i.i.d. random variables distributed uniformly*

in a disk of radius $r < 1$, and α_{n-1} is another random variable independent of the previous ones and uniformly distributed on the unit circle.

Consider the space $\Omega_n = \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}) \in D(0,r) \times D(0,r) \times \cdots \times D(0,r) \times \mathbb{T}\}$ with the probability measure $\mathbb{P}_n$ obtained by taking the product of the uniform (Lebesgue) measures on each $D(0,r)$ and on $\mathbb{T}$. Fix a point $e^{i\theta_0} \in \mathbb{T}$ and let $\zeta^{(n)}$ be the point process defined by $\zeta^{(n)} = \sum_{k=1}^{n} \delta_{z_k}$, where $\{z_1, z_2, \dots, z_n\}$ are the eigenvalues of the matrix $\mathcal{C}^{(n)}$ (each eigenvalue z_i depends on $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$).

Then, on a fine scale (of order $\frac{1}{n}$) near $e^{i\theta_0}$, the point process $\zeta^{(n)}$ converges to the Poisson point process with intensity measure $n\frac{d\theta}{2\pi}$ (where $\frac{d\theta}{2\pi}$ is the normalized Lebesgue measure). This means that for any fixed $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ and any nonnegative integers $k_1, k_2, \dots, k_m$, we have

$$\mathbb{P}_n\left(\zeta^{(n)}\left(e^{i(\theta_0+\frac{2\pi a_1}{n})}, e^{i(\theta_0+\frac{2\pi b_1}{n})}\right) = k_1, \dots, \ \zeta^{(n)}\left(e^{i(\theta_0+\frac{2\pi a_m}{n})}, e^{i(\theta_0+\frac{2\pi b_m}{n})}\right) = k_m\right)$$
$$\longrightarrow e^{-(b_1-a_1)}\frac{(b_1-a_1)^{k_1}}{k_1!} \ \dots \ e^{-(b_m-a_m)}\frac{(b_m-a_m)^{k_m}}{k_m!} \quad (3.5)$$

as $n \to \infty$.

The result is similar to the ones obtained for random Schrödinger operators by Molchanov and Minami (see Section 2). All three proofs use the same road map towards obtaining the Poisson distribution in the limit:

1. uniform estimates on the resolvent
2. control of the eigenfunctions
3. decoupling of the random matrix into smaller identical blocks
4. proving that the smaller blocks cannot contribute two or more eigenvalues in given intervals
5. obtaining the Poisson distribution as a limit of Bernoulli distributions

We will give a few more details about this road map, following [21]. For the random CMV matrices considered in Theorem 3.1, we can prove that there exist two constants C and D such that for any $z \in \mathbb{C}$, $|z| < 1$ and any $s < 1$,

$$\mathbb{E}\left(\left|F_{kl}(z, \mathcal{C}_\alpha^{(n)})\right|^s\right) \leq Ce^{-D|k-l|} \quad (3.6)$$

where

$$F_{kl}(z, \mathcal{C}_\alpha^{(n)}) = \left[\frac{\mathcal{C}_\alpha^{(n)} + z}{\mathcal{C}_\alpha^{(n)} - z}\right]_{kl} \quad (3.7)$$

is the entry in row k and column l of the matrix $\left[\frac{\mathcal{C}_\alpha^{(n)}+z}{\mathcal{C}_\alpha^{(n)}-z}\right]$. This idea (of controlling fractional moments of the resolvent) originates from [3].

The steps for getting the exponential decay of $\mathbb{E}\left(\left|F_{kl}(z,\mathcal{C}_\alpha^{(n)})\right|^s\right)$ (relation (3.6)) are:

a) Uniform upper bound for all the fractional moments:

$$\mathbb{E}\left(\left|F_{kl}(z,\mathcal{C}_\alpha^{(n)})\right|^s\right) \leq \frac{2^{2-s}}{\cos\frac{\pi s}{2}} \tag{3.8}$$

b) Uniform decay on rows: For every $\varepsilon > 0$ there exists a $k_\varepsilon > 0$ such that for any $k \geq k_\varepsilon$ and for any j, we have

$$\mathbb{E}\left(\left|F_{j,j+k}(z,\mathcal{C}_\alpha^{(n)})\right|^s\right) \leq \varepsilon \tag{3.9}$$

c) Spectral averaging estimate (i.e. evaluating the conditional expectation using a finite rank perturbation)

$$\mathbb{E}\left(\left|F_{kk}(z,\mathcal{C}_\alpha^{(n)})\right|^s \;\middle|\; \{\alpha_i\}_{i\neq k}\right) \leq \frac{4}{1-s}\,32^s \tag{3.10}$$

d) Uniform decay implies uniform exponential decay

$$\mathbb{E}\left(\left|F_{kl}(z,\mathcal{C}_\alpha^{(n)})\right|^s\right) \leq Ce^{-D|k-l|} \tag{3.11}$$

for some positive constants $C, D > 0$. This follows from a method developed by Aizenman et al. which shows that fast power decay implies exponential decay.

The bounds (3.6) can be used to prove that the eigenfunctions of the random matrix $\mathcal{C}_\alpha^{(n)}$ are exponentially localized with probability 1. We should mention that in this step we use Simon's extension to CMV matrices (see [17]) of a result of Aizenman on dynamical localization for Schrödinger operators (see [1]). We should mention that this is one of the places where our assumption that the common distribution of the Verblunsky coefficients α_k is rotation invariant is critical.

The exponential localization allows us to decouple the CMV matrix $\mathcal{C}^{(n)} = \mathcal{C}_\alpha^{(n)}$. More precisely, we can consider another unitary matrix $\tilde{\mathcal{C}}^{(n)} = \tilde{\mathcal{C}}_\alpha^{(n)}$ which decouples into the sum of $[\ln n]$ smaller identical unitary matrices:

$$\tilde{\mathcal{C}}^{(n)} = \tilde{\mathcal{C}}_1^{(n)} \oplus \tilde{\mathcal{C}}_2^{(n)} \oplus \cdots \oplus \tilde{\mathcal{C}}_{[\ln n]}^{(n)} \tag{3.12}$$

such that the statistical distribution of the eigenvalues of the matrix $\mathcal{C}^{(n)}$ is the same as the statistical distribution of the eigenvalues of the matrix $\tilde{\mathcal{C}}^{(n)}$.

For any p, $1 \leq p \leq [\ln n]$, we consider the point process $\zeta^{(n,p)} = \sum_{k=1}^{[\ln n]} \delta_{z_k^{(p)}}$, where $z_1^{(p)}, z_2^{(p)}, \ldots, z_{[n/\ln n]}^{(p)}$ are the eigenvalues of the matrix $\tilde{\mathcal{C}}_p^{(n)}$. Therefore $\zeta^{(n)}$ can be approximated by $\sum_{k=1}^{[\ln n]} \zeta^{(n,p)}$. This means that the point process $\zeta^{(n)}$ is infinitely divisible as $n \to \infty$. References for the general theory of point processes are [6] and [11].

The next step is to prove that each small matrix $\tilde{\mathcal{C}}_p^{(n)}$ contributes at most one eigenvalue in the interval of size $\frac{1}{n}$. This follows from the fact that for any $e^{i\theta} \in \mathbb{T}$ and any interval I_n of size $\frac{1}{n}$ near $e^{i\theta}$ we have

$$\mathbb{P}_n \left(\zeta^{(n,p)}(I_n) \geq 2 \right) = O([\ln n]^{-2}) \quad \text{as} \quad n \to \infty \tag{3.13}$$

which implies

$$\sum_{p=1}^{[\ln n]} \mathbb{P}_n \left(\zeta^{(n,p)}(I_n) \geq 2 \right) \longrightarrow 0 \quad \text{as} \quad n \to \infty \tag{3.14}$$

Hence an interval of size $\frac{1}{n}$ contains k eigenvalues if and only if exactly k of the matrices $\tilde{\mathcal{C}}_1^{(n)}, \tilde{\mathcal{C}}_2^{(n)}, \ldots, \tilde{\mathcal{C}}_{[\ln n]}^{(n)}$ contribute one eigenvalue in the interval I_n. This is exactly a Bernoulli distribution and its limit is a Poisson distribution. Hence we conclude that the limit $\zeta^{(n)}$ is a Poisson point process.

Theorem main shows that if α_k, $0 \leq k \leq (n-2)$ are uniformly distributed on a disk of radius $r < 1$, then the eigenvalues of the corresponding CMV matrix $\mathcal{C}^{(n)}$ are not locally correlated for large n. A natural question is whether this result can be extended to different types of distributions for the Verblunsky coefficients α_k. It turns out that the rotation invariance was critical in several places, but we can allow distributions that are singular with respect to the Lebesgue measure. Thus, if for $r \in (0,1)$ we denote by $C(0,r)$ the circle centered at the origin and of radius r, we get:

Theorem 3.2. *Consider the random CMV matrices* $\mathcal{C}^{(n)} = \mathcal{C}^{(n)}_{\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}}$ *where* $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ *are i.i.d. random variables distributed uniformly in a circle of radius* $r < 1$*, and* α_{n-1} *is another random variable independent of the previous ones and uniformly distributed on the unit circle.*

Consider the space $\Omega_n = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}) \in C(0,r) \times C(0,r) \times \cdots \times C(0,r) \times \mathbb{T}\}$ *with the probability measure* $\mathbb{P}_n$ *obtained by taking the product of the uniform (one-dimensional Lebesgue) measures on each* $C(0,r)$ *and on* $\mathbb{T}$*. Fix a point* $e^{i\theta_0} \in \mathbb{T}$ *and let* $\zeta^{(n)}$ *be the point process obtained from the eigenvalues of the truncated CMV matrix* $\mathcal{C}^{(n)}$*.*

Then, on a fine scale (of order $\frac{1}{n}$) near $e^{i\theta_0}$, the point process $\zeta^{(n)}$ converges to the Poisson point process with intensity measure $n\,\frac{d\theta}{2\pi}$ (where $\frac{d\theta}{2\pi}$ is the normalized Lebesgue measure). This means that for any fixed $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ and any nonnegative integers $k_1, k_2, \ldots, k_m$, we have

$$\mathbb{P}_n\left(\zeta^{(n)}\left(e^{i(\theta_0+\frac{2\pi a_1}{n})}, e^{i(\theta_0+\frac{2\pi b_1}{n})}\right) = k_1, \ldots,\ \zeta^{(n)}\left(e^{i(\theta_0+\frac{2\pi a_m}{n})}, e^{i(\theta_0+\frac{2\pi b_m}{n})}\right) = k_m\right)$$
$$\longrightarrow e^{-(b_1-a_1)}\frac{(b_1-a_1)^{k_1}}{k_1!}\cdots e^{-(b_m-a_m)}\frac{(b_m-a_m)^{k_m}}{k_m!} \quad (3.15)$$

as $n \to \infty$.

Proof. The proof is similar to the proof of the Theorem 3.1, with a some modifications required by the fact that we use a different probability distribution for the random Verblunsky coefficients.

With the exception of (3.10), all the steps outlined before hold when we replace the uniform distribution on the disk or radius r with the uniform distribution on the circle of radius r.

Lemma 3.3 (see below) shows that the conditional moments of order s with $s \in (0, \frac{1}{2})$ are uniformly bounded. A standard application of Hölder's theorem implies that the result in Lemma 3.3 holds for all $s \in (0, 1)$.

We should also mention that Aizenman's theorem for orthogonal polynomials on the unit circle (see [17]) holds for the distribution Verbunsky coefficients uniformly distributed on the circle of radius r. As in the proof of Theorem 3.1, we can now derive the exponential decay of the fractional moments of the resolvent of $\mathcal{C}_\alpha^{(n)}$.

Once we have the exponential decay of the fractional moments of the matrix elements of the resolvent of $\mathcal{C}_\alpha^{(n)}$, we can follow the same route as in the proof of Theorem 3.1 (the eigenfunctions are exponentially localized, the matrix $\mathcal{C}_\alpha^{(n)}$ can be decoupled, and the decoupled matrices contribute at most one eigenvalue in each interval of size $O(\frac{1}{n})$) to conclude that (3.15) holds, that is, the local statistical distribution of the eigenvalues of $\mathcal{C}_\alpha^{(n)}$ is Poisson.

□

We will now give the analog of (3.10) for the case of Verblunsky coefficients uniformly distributed on $C(0, r)$.

Lemma 3.3. *For any $s \in (0, \frac{1}{2})$, any k, $1 \leq k \leq n$, and any choice of $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ as in Theorem 3.2,*

$$\mathbb{E}\left(\left|F_{kk}(z, \mathcal{C}_\alpha^{(n)})\right|^s \;\middle|\; \{\alpha_i\}_{i\neq k}\right) \leq K(s, r) \quad (3.16)$$

where a possible value for the constant is $K(s,r) = 4 \cdot 64^s \cdot \frac{\pi^{1-2s}}{(1-2s)r^s}$.

Proof. Using properties of orthogonal polynomials on the unit circle (see [9], [12] and also [21] and [22]), the dependence of $F_{kk}(z, \mathcal{C}_\alpha^{(n)})$ on α_k is of the form

$$F_{kk}(z, \mathcal{C}_\alpha^{(n)}) = \left| \frac{2}{1 - w_1 \frac{\alpha_k + w_2}{1 + \overline{\alpha_k} w_2}} \right| \tag{3.17}$$

where $w_1, w_2 \in \mathbb{C}$ are complex numbers of absolute value strictly less than 1 that do not depend on α_k.

Therefore, in order to prove (3.16), it is enough to find a uniform bound for

$$I = \int_{C(0,r)} \left| \frac{4}{1 + \overline{\alpha}_k w_2 - w_1(\alpha_k + w_2)} \right|^s d\mu_r(\alpha_k) \tag{3.18}$$

for any $w_1, w_2 \in \mathbb{D}$, (μ_r denotes one-dimensional Lebesgue measure $\frac{d\theta}{2\pi}$ on $C(0,r)$).

Let $\alpha_k = x + iy$, with $x, y \in \mathbb{R}$. Then

$$1 + \overline{\alpha}_k w_2 - w_1(\alpha_k + w_2) = x(-w_1 + w_2) + y(-iw_1 - iw_2) + (1 - w_1 w_2) \tag{3.19}$$

Let's denote by $M(x, y, w_1, w_2)$ the right-hand-side of the previous equation.

Then

$$\int_{C(0,r)} \left| \frac{4}{M(x, y, w_1, w_2)} \right|^s d\mu_r(\alpha_k) \leq \min\{I_1, I_2\} \tag{3.20}$$

where

$$I_1 = \int_{C(0,r)} \left| \frac{4}{\text{Re } M(x, y, w_1, w_2)} \right|^s d\mu_r(\alpha_k) \quad \text{and} \tag{3.21}$$

$$I_2 = \int_{C(0,r)} \left| \frac{4}{\text{Im } M(x, y, w_1, w_2)} \right|^s d\mu_r(\alpha_k) \tag{3.22}$$

Let $\varepsilon > 0$ (we will choose ε later). If $|w_2 - w_1| < \varepsilon$ and $|w_2 + w_1| < \varepsilon$, then $|M(x, y, w_1, w_2)| \geq 1 - 2\varepsilon - \varepsilon^2$. We immediately get

$$\int_{C(0,r)} \left| \frac{4}{M(x, y, w_1, w_2)} \right|^s d\mu_r(\alpha_k) \leq \left(\frac{4}{1 - 2\varepsilon - \varepsilon^2} \right)^s \tag{3.23}$$

If at least one of the complex numbers $(w_2 - w_1)$ and $(w_2 + w_1)$ is greater or equal in absolute value than ε, then at least one of

$$M_1(w_1, w_2) = \sqrt{(\text{Re}(w_2 - w_1))^2 + (\text{Re}(-iw_1 - iw_2))^2} \quad \text{and} \tag{3.24}$$

$$M_2(w_1, w_2) = \sqrt{(\text{Im}(w_2 - w_1))^2 + (\text{Im}(-iw_1 - iw_2))^2} \tag{3.25}$$

is greater than $\varepsilon/2$. Without loss of generality, we can assume that $M_1(w_1, w_2) \geq \varepsilon/2$.

Then

$$I_1 = \int_{C(0,r)} \left| \frac{4}{x \operatorname{Re}(-w_1 + w_2) + y \operatorname{Re}(-iw_1 - iw_2) + \operatorname{Re}(1 - w_1 w_2)} \right|^s d\mu_r(\alpha_k) \tag{3.26}$$

$$= \int_{C(0,r)} \left| \frac{4}{x \frac{\operatorname{Re}(-w_1+w_2)}{M_1(w_1,w_2)} + y \frac{\operatorname{Re}(-iw_1-iw_2)}{M_1(w_1,w_2)} + \frac{\operatorname{Re}(1-w_1w_2)}{M_1(w_1,w_2)}} \right|^s \left| \frac{1}{M_1(w_1, w_2)} \right|^s d\mu_r(\alpha_k) \tag{3.27}$$

$$= \int_0^{2\pi} \frac{1}{r^s} \left| \frac{4}{\cos\theta \frac{\operatorname{Re}(-w_1+w_2)}{M_1(w_1,w_2)} + \sin\theta \frac{\operatorname{Re}(-iw_1-iw_2)}{M_1(w_1,w_2)} + \frac{\operatorname{Re}(1-w_1w_2)}{rM_1(w_1,w_2)}} \right|^s \left| \frac{1}{M_1(w_1, w_2)} \right|^s \frac{d\theta}{2\pi} \tag{3.28}$$

Let $\theta_0 = \theta_0(w_1, w_2)$ be an angle such that $\sin\theta_0 = \frac{\operatorname{Re}(-w_1+w_2)}{M_1(w_1,w_2)}$ and $\cos\theta_0 = \frac{\operatorname{Re}(-iw_1-iw_2)}{M_1(w_1,w_2)}$ and let $K_0 = K_0(w_1, w_2, r)$ be defined by $K_0 = \frac{\operatorname{Re}(1-w_1w_2)}{rM_1(w_1,w_2)}$.

Using (3.28), we get

$$I_1 \leq \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{\sin(\theta + \theta_0) + K_0} \right|^s \frac{d\theta}{2\pi} = \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{\sin\theta + K_0} \right|^s \frac{d\theta}{2\pi} \tag{3.29}$$

Without loss of generality, we can assume that in the previous relation we have $K_0 \in [-1, 1]$ and therefore $K_0 = \sin\kappa$, for a $\kappa \in [0, 2\pi)$. Therefore,

$$I_1 \leq \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{2\left(\sin(\frac{\theta+\kappa}{2})\cos(\frac{\theta-\kappa}{2})\right)} \right|^s \frac{d\theta}{2\pi} = \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{2\left(\sin(\frac{\theta+\kappa}{2})\sin(\frac{\pi-\theta+\kappa}{2})\right)} \right|^s \frac{d\theta}{2\pi} \tag{3.30}$$

The function $\sin x$ vanishes as fast as x at each zero. We actually have

$$|\sin x| \geq \frac{2}{\pi}|x| \qquad \text{for each } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{3.31}$$

so using (3.30) and the fact that the function sin vanishes twice in an interval of length 2π, we get

$$I_1 \leq 4 \cdot \frac{4^s}{\varepsilon^s r^s} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{|\theta|^{2s}} \frac{d\theta}{2\pi} = \frac{2^{2s+3}}{\varepsilon^s r^s} \frac{\left(\frac{\pi}{2}\right)^{1-2s}}{1-2s} = 2^{4s+2} \frac{\pi^{1-2s}}{\varepsilon^s r^s} \frac{1}{1-2s} \tag{3.32}$$

We can therefore conclude that for I defined in (3.18), we have

$$I \leq \max\left\{\left(\frac{4}{1-2\varepsilon-\varepsilon^2}\right)^s, 2^{4s+2}\,\frac{\pi^{1-2s}}{\varepsilon^s r^s}\,\frac{1}{1-2s}\right\} \tag{3.33}$$

For $\varepsilon = \frac{1}{4}$ we get

$$I \leq 4 \cdot 64^s \cdot \frac{\pi^{1-2s}}{r^s(1-2s)} \tag{3.34}$$

which is exactly (3.16). □

Theorems 3.1 and 3.2 show that if the Verblunsky coefficients are independent and identically distributed, then the statistical distribution of the eigenvalues of the random CMV matrices obtained is Poisson (no correlation). Recent work of Simon [18] showed that if the Verblunsky coefficients are independent random variables and with exponentially decaying distributions, then the eigenvalues of the CMV matrices repel each other giving a "clock distribution" on the unit circle. It would be very interesting to discover which distributions of the random Verblunsky coefficients give a transition in the statistical distribution of the eigenvalues from non-correlation (Poisson) to repulsion ("clock").

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