WCMC Math Challenge Questions - Summer 2017

Instructions. Try as many of the following questions as you can. There is no time limit, but you should do these questions **on your own without any help from people, books, internet, or any other sources**. We are more interested in how you approach the questions and how you communicate your reasoning than in how many correct answers you obtain. For each solution you submit, please include a clear and complete explanation of your answer. Please typeset your answers. Send any questions you have to Allison.Pacelli@williams.edu.

1. There's a small village at the base of a mountain with a population of 8000. There are at least 2 people in the village that do not have dogs. Also, given any three people in the village, at least one of the three has a dog. Do we have enough information to determine exactly how many people in the village have dogs? Explain why not, or find the exact number. Explain your reasoning.

2. Let m and n be positive integers. If m has a digits and n has b digits, what are the possibilities for the number of digits that the product mn has? Explain your answer completely.

3. Find all integers M that satisfy all three of the following properties. Explain your answer fully.

(i) If 2 divides M, then $40 \leq M \leq 70$.

(ii) If 6 does not divide M, then $40 \leq M \leq 57$.

(iii) 9 divides M.

4. Characterize completely the set of all such prime numbers p and q for which $pq + 1$ is a perfect square. Prove that your characterization is correct.

5. There's a box of five hats: two blue and three white. Andy, Kate, and James each place a hat on his or her head, while blindfolded. One by one, each child removes his blindfold and (without using a mirror) gets one opportunity to guess the color of the hat on his own head. If any of the three guesses correctly, everyone gets to go to the park! Andy, Kate, and James are each very logical, and know that the others are as well.

First, James removes his blindfold. He sees the hats that the others are wearing, but admits that he is unable to discern his own hat color.

Next, Kate removes her blindfold, and sadly reveals that she too is not able to determine the color of her own hat.

Finally, Andy pipes up and says "I can answer with my blindfold on! I know what color hat

I am wearing."

What color is Andy's hat, and how does he know? Explain fully.

6. Andy encounters a strange island, where every creature has either green, purple, or blue hair on his head. He's told by a reliable source that those with green hair always tell the truth, those with purple hair always lie, and those with blue hair make statements that are alternately true and false (though the order of which statements are true and which are false is unknown). One day, Andy meets three of the islanders, but each is wearing a very big hat which completely covers his hair. Andy asks each of the three the color of their hair, and the responses are as follows:

A: I have green hair.

B: I have purple hair.

C: 1. They are both lying. 2. I have blue hair.

Assuming that each of the three has a different color hair, determine the hair color of each of A, B, and C. Explain your answer fully. If we don't assume that each has a different hair color, is it possible to determine the hair color of each islander? Explain fully.

7. You are blindfolded, and on the table in front of you are a number of shiny smooth discs. They are each about the size of a quarter; one side is purple and the other side green. You can certainly determine how many discs there are altogether, but you can't tell whether a given disc is purple side up or green side up. Your friend Michael tells you (once) how many are purple side up. Your challenge is to separate the discs into two collections, flipping over whichever discs you like, so that each collection has the same number of purple side up discs. Explain how you complete the challenge, and how you know you're successful.

8. Write down a mathematically interesting extension to one of the problems above that one could investigate. Do not do the problem, just pose the question.