The Honeycomb Problem on Hyperbolic Surfaces

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1 Introduction

In 1999 Thomas Hales proved the Hexagonal Honeycomb Conjecture [2], which states that regular hexagons provide the most efficient partition of the plane into equal areas. A key element in the proof is the hexagonal isoperimetric inequality, which compares the perimeter of an arbitrary curvilinear polygon to the perimeter of a regular hexagon. Hales later applied similar techniques to the unit sphere, using a pentagonal isoperimetric inequality to show that a tiling of twelve regular pentagons in a dodecahedral arrangement is the unique minimizer [3]. By applying some of the lower bounds used by Hales to hyperbolic surfaces, polygonal isoperimetric inequalities can be used to prove that a valence three tiling of regular heptagons and octagons on a compact hyperbolic manifold is perimeter minimizing.

2 Perimeter Minimizing Partitions of Compact Hyperbolic Surfaces

In contrast to the Euclidean plane, the hyperbolic plane can be partitioned with regular N-gons for $N = 7$ or $N = 8$ and using any valence q for the vertices. If $q > 3$ then the perimeter can be reduced locally at the vertices; if $q = 3$ then we will show that the partition is minimal.

Let M be a closed surface and suppose V, E, and F satisfy the requirement $V - E + F = \mathcal{X}$ for the Euler characteristic $\mathcal{X}(M)$. Then for any $N \geq 7$ such that $NF = 2E = 3V$ there exists a tessellation of M with F N-sided faces, E edges, and V valence three vertices. Furthermore, M admits a Riemannian metric of constant curvature -1 under which all of the edges are geodesics of equal length and all of the interior angles are $\frac{2\pi}{3}$. (See Edmonds [1].) If F_N is the number of faces in the tessellation of regular N-sided polygons then

$$\mathcal{X} = F_N(1 - \frac{N}{6}).$$

Lemma 2.1. *Let $\{P_i\}$ be a partition of a closed surface of Euler characteristic $\mathcal{X}$ into at least $\mathcal{X} = F_N(1 - \frac{N}{6})$ regions where $N \geq 7$ is an integer and each* *region P_i is a union of curvilinear polygons with $N(P_i)$ edges. Then*

$$\sum(N(P_i) - N) \leq 0.$$

Equality is achieved only for the partition of regular polygons.

Let $\hat{N} = 7$ or 8 and let M be a compact hyperbolic manifold of constant curvature that admits a partition of regular $\hat{N}$-gons. Let P be a polygon on M with $N = N(P)$ vertices, perimeter $L(P)$, and area $A = A(P)$. For polygonal components of regions partitioning M, we may assume that $A(P) \geq \frac{2\pi}{\sqrt{3}N^2}$. Given a smaller region we can delete edges to create a new partition with smaller total perimeter. [See [2] remark 2.7.] Also, we may assume $A(P) \leq \frac{\hat{N}-6}{3}\pi$, the total area of a region determined by $\hat{N}$.

Index the edges e_i of P and define $x(e_i)$ to be the signed area between e_i and the chord sharing the same vertices. Let τ be the truncation function

$$\tau(x) = \begin{cases} \frac{1}{2}, & x \geq \frac{1}{2} \\ x, & |x| < \frac{1}{2} \\ -\frac{1}{2}, & x \leq -\frac{1}{2} \end{cases}$$

Let $T(P) = \sum_{i=1}^{N} \tau(x(e_i))$.

Lemma 2.2. The Polygonal Isoperimetric Inequality *Let P be a curvilinear polygon on M with $\frac{2\pi}{\sqrt{3}N^2} \leq A(P) \leq \frac{N-6}{3}\pi$. Then*

$$L(P) \geq \frac{3P_N}{(N-6)\pi} A(P) - bT(P) - c(N(P) - N)$$

where P_N is the perimeter of a regular N-gon of area A(P), $b = 1.8$, and $c = .05$.

The proof of this lemma is featured prominently in the next section.

Theorem 2.3. *The partition consisting of regular N-gons is the least perimeter way to partition M into regions of area $\frac{N-6}{3}\pi$.*

The Euler characteristic of M will uniquely determine the number of regions in the partition.

Proof. Let $\{R_i\}$ be a partition of M into equal area regions of area $\frac{N-6}{3}\pi$. We may assume that each region is the finite union of curvilinear polygons, $R_i = \cup P_i$, with vertices meeting in threes. (See [4].) Each P_i satisfies the area requirement of Lemma 2.2.

Each edge e_i of P_i occurs with opposite orientation as an edge e_j on an adjacent region P_j, therefore the area $x(e_i)$ is identical in magnitude and opposite in sign to $x(e_j)$. Summing over all the edges of the partition gives $\sum T(P_i) = 0$.

Applying the polygonal isoperimetric inequality and 2.1, twice the total perimeter of the partition is bounded by

$$\sum L(P_i) \geq \frac{3P_N}{(N-6)\pi} \sum A(P) - 2\sum T(P) - .05\sum(N(P) - N)$$

$$= \frac{3P_N}{(N-6)\pi} \sum A(P)$$

$$= \frac{3P_N}{(N-6)\pi} F_N \frac{N-6}{3} \pi.$$

Thus the perimeter is at least $P_N F_N$, the length of the perimeter of the regular partition. □

3 Proof of the Hyperbolic Polygonal Isoperimetric Inequality

To prove Lemma 2.2 it is sufficient to demonstrate that it holds for all possible values of $N(P)$, $T(P)$, and all $A(P) < \frac{N-6}{3}\pi$.

Let $F(N,T,A) = \frac{3P_N}{(N-6)\pi} A(P) - 2T(P) - .05(N(P) - \hat{N})$ so that $\Delta P = L(P) - F$ and it suffices to show that $\Delta P \geq 0$ for any number of sides N, all values of T, and any admissible area A. By the definition of N and T we know immediately that $N \geq 2|T|$ and $N \geq 2$.

First we establish some useful lower bounds for the perimeter L(P) of a curvilinear polygon as function of the number of sides N and the enclosed area A. The prove then considers three cases, the digon or $N = 2$ case, the region of the NT-plane covered by bounds L_- and L_N, and a small portion of the plane that requires the bound L'_N.

3.1 Lower Bounds

$\mathbf{L_N}$: The perimeter is at least as big as L_N, the perimeter of a regular N-gon.

$$P \geq L_N = 2N cosh^{-1} \left(\frac{cos \frac{\pi}{N}}{sin \left(\frac{\pi(N-2)-A}{2N} \right)} \right)$$

$\mathbf{L_+}$: The perimeter is at least as big as the circumference of a circle enclosing the same area.

$$P \geq L_+ = \sqrt{A(4\pi + A)}$$

$\mathbf{L_-}$: The perimeter is at least as great as the circumference of a circle having the area of the modified polygon obtained by reflecting each inward curving edge outward across the chord.

Let J be the set of indices of the edges that curve inward, that is, the set of edges with $x(e_j) \leq 0$ and $X_J = \sum_{j \in J} x(e_j)$.

$$P \geq L_- = \sqrt{(A - 2X_J)(4\pi + A - 2X_J)}$$

Note: This bound does not seem especially useful. $\mathbf{L_D}$: According to Dido a semicircle enclosing a given area has a smaller perimeter than any other arc of a circle enclosing the same area. Applying this to all the edges and letting $X_D = \sum |x(e)|$ we have another lower bound.

$$P \geq L_D = X_D\sqrt{4\pi + 1}$$

$\mathbf{L'_N}$: If all the chords have length at most 1 and $| \sum x_i | \leq 4\pi sinh^2(\frac{1}{4})$ the the perimeter is at least L_N times the length of a circular arc of chord 1 enclosing the area $\frac{|\sum x_i|}{L_N}$.

3.2 N(P)=2

Suppose that $N(P) = 2$ so that P is a digon. Then $A(P) = x(e_1) + x(e_2) \geq A_{min} = \frac{\pi}{2\sqrt{3}} > .9$.

First consider the case $\hat{N} = 7$ where $b = 1.8$. If $T \geq \frac{1}{4}$, then the inequality follows from the bound L_+. Since $N = 2$ and ΔP is an increasing function of T,

$$\Delta P \geq L_+ - F(2, \frac{1}{4}, A)$$

$$= \sqrt{A(4\pi + A)} - \frac{3}{\pi}P_7 A + \frac{1}{4}b + c(2 - 7)$$

$$= \sqrt{A(4\pi + A)} - \frac{3}{\pi}P_7 A + .2 > 0$$

if $A \in [.9, \frac{\pi}{3}]$.

If $T < \frac{1}{4}$ we use the bound L_-. Since $A = x_1 + x_2 \geq .9$, $x_{max} > \frac{1}{2}$ and

$$T = x_{min} + \frac{1}{2} < \frac{1}{4}$$

$$x_{min} < -\frac{1}{4}.$$

Applying the lower bound L_-, in this case with the single negative area x_{min},

$$\Delta P \geq L_- - F(2, T, A)$$

$$\geq \sqrt{(A - 2x_{min})(4\pi + A - 2x_{min})} - \frac{3}{\pi}P_7 A + 1.8(x_{min} + \frac{1}{2}) + c(2 - 7) > 0$$

for $A \in [.9, \frac{\pi}{3}]$, $x_{min} \in [-\frac{1}{2}, -\frac{1}{4}]$.

The case $\hat{N} = 8$ follows similarly. *However, this argument cannot be automatically generalized for any $\hat{N}$ because the range covered by L_+ decreases, the cutoff value for T increasing. This can be countered by choosing a value for b that increases, such as $b = (9/25)(3\hat{N} - 16)$, but L_- becomes difficult if $b > 2$.*

3.3 N(P)≥3, L_N and L_-

For $N(P) \geq 3$ the bounds L_N and L_- ensure that the inequality holds everywhere except a tiny region with which we almost neednt concern ourselves. (Since N takes on only integer values, the region is more accurately a collection of intervals in T for a finite set of N values. Setting $L_- - F(N,T,A) = 0$ for a fixed area yields a curve in the NT-plane,

$$(b^2-4)T^2+(2cbN-2abA-2bc\hat{N}+8\pi+4A)T+(a^2-1)A^2+2acA\hat{N}+c^2\hat{N}^2-4\pi A = 0.$$

The inequality holds for all values of T below the curve, and for a fixed N the value of T increases as A increasing. Hence, we need only consider the extreme case where $A = A_{max}$.

Similarly, the bound L_N generates the bounding curve

$$T = \frac{2N cosh^{-1} \frac{cos\frac{\pi}{2}}{sin\frac{\pi(N-2)-A}{2N}} - \frac{3P_N}{(N-6)\pi}A + c(N - \hat{N}}{-b}.$$

The inequality holds for all T above the bound, including all $T > 0$ for the correct constants b and c, and again we need only consider the extreme case of maximal area. (For sufficient small area the two boundaries completely overlap and the chordal bound is not needed.)

3.4 The Chordal Bound

Finally, we directly verify the following intervals using the chordal bound.

$\hat{N} = 7$
$N = 5 : [-.064, -.153]$
$N = 6 : [-.014, -.124]$
$N = 7 : [0, -.097]$
$N = 8 : [-.001, -.070]$
$N = 9 : [-.012, -.044]$
$\hat{N} = 8$
$N = 5 : [-.217, -.397]$
$N = 6 : [-.083, -.345]$
$N = 7 : [-.024, -.294]$
$N = 8 : [.001, -.245]$
$N = 9 : [.007, -.198]$
$N = 10 : [.003, -.152]$
$N = 11 : [-.008, -.107]$
$N = 12 : [-.023, -.063]$

References

[1] A. L. Edmonds, J. H. Ewing, R. S. Kulkarni, *Regular Tessellations of Surfaces and (p,q,2)-triangle Groups*, The Annals of Mathematics, Second Series, Volume 116, Issue 1 (July, 1982), 113-132.

[2] T. Hales, *The Honeycomb Conjecture*, Discr. Comput. Geom. 25:1-22 (2001).

[3] T. Hales, *The Honeycomb Problem on the Sphere*, (2002).

[4] F. Morgan, *Soap Bubbles in* $\Re^2$ *and in Surfaces*, Pacific J. Math, 165 (1994), no. 2, 347-361.

[5] F. Morgan, *Geometric Measure Theory*, Academic Press, San Diego, 3rd Ed., 2000.