The Honeycomb Problem on Hyperbolic Surfaces

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1 Introduction

In 1999 Thomas Hales proved the Hexagonal Honeycomb Conjecture [2], which states that regular hexagons provide the most efficient partition the plane into equal areas. A key element in the proof is the hexagonal isoperimetric inequality, which compares the perimeter of an arbitrary curvilinear polygon to the perimeter of a regular hexagon. Hales later applied the same method to the unit sphere, using a pentagonal isoperimetric inequality to show that a tiling of twelve regular pentagons in a dodecahedral arrangement is the unique minimizer [3]. By applying the techniques to hyperbolic surfaces, polygonal isoperimetric inequalities can be used to prove that if $7 \leq N \leq 16$ a tiling of regular N-gons on a compact hyperbolic manifold is perimeter minimizing.

In contrast to the Euclidean plane, the hyperbolic plane can be partitioned with regular N-gons for any $N \geq 7$ and using any valence q for the vertices. In particular, suppose M is a closed surface and V, E, and F satisfy the requirement

$$V - E + F = X(M)$$

for the Euler characteristic $\mathcal{X}(M)$. Then for any $N \geq 7$ and $q \geq 3$ such that $NF = 2E = qV$ there exists a tessellation on M with F N-sided faces, E edges, and V valence q vertices. Furthermore, M admits a Riemannian metric of constant curvature -1 under which all of the edges are geodesics of equal length and all of the interior angles are $\frac{2\pi}{q}$. (See [1].)

2 The Heptagonal Partition of a Genus 3 Orientable Surface

Consider the case where $N = 7$ and $q = 3$ on the manifold M, a genus 3 orientable surface. The regular heptagonal partition has 24 faces, 84 edges, and 56 vertices. Under the hyperbolic metric of constant curvature -1 the area of each heptagon is $\frac{\pi}{3}$ and $P_7 \approx 3.96$ is the perimeter.

The fact that this partition is length minimizing will follow from the following lemmas.

Lemma 2.1. *Let $\{P_i\}$ be a partition of M into at least 24 regions, where each P_i is a curvilinear polygon with $N(P_i)$ edges. Then*

$$\sum (N(P_i) - 7) \leq 0.$$

Equality is achieved only for the regular heptagonal partition.

Proof. Let F, E, and V represent the number of faces, edges, and vertices in the given partition, viewing all vertices as having valence three by inserting edges of length zero if necessary.

$$\begin{aligned}\sum (N(P_i) - 7) &= \sum (N(P_i)) - 7 \sum 1 \\ &= 6 \sum \left(\frac{N(P_i)}{2} - \frac{N(P_i)}{3} - 1 \right) - \sum 1 \\ &= 6(E - V - F) - F \\ &= 6(X(M)) - F \\ &= 24 - F \leq 0.\end{aligned}$$

□

Let P be a curvilinear polygon with $N(P)$ vertices on M with perimeter $L(P)$ and area $A(P)$. Index the edges e_i and define $x(e_i)$ to be the signed area between e_i and the chord sharing the same vertices. Let τ be the truncation function

$$\tau(x) = \begin{cases} \frac{1}{2}, & x \geq \frac{1}{2} \\ x, & |x| < \frac{1}{2} \\ \frac{1}{2}, & x \leq -\frac{1}{2} \end{cases}$$

Let $T(P) = \sum_{i=1}^{N} \tau(x(e_i))$.

Lemma 2.2. *Let P be a curvilinear polygon in the hyperbolic plane with $\alpha(N(P)) = \frac{2\pi}{\sqrt{3}N^2} < A(P) < \frac{\pi}{3}$. Then*

$$L(P) \geq \frac{3}{\pi} P_7 A(P) - T(P) \frac{P_7}{2} - .112(N(P) - 7)$$

The proof is given in the next section.

Remark 2.1. We may assume that every N-sided component of a partition has area of at least $\alpha(N) = \frac{2\pi}{\sqrt{3}N^2}$. Given a smaller region we can delete edges to create a new partition with smaller total perimeter. [See Hales remark 2.7.]

Theorem 2.3. *The regular heptagonal partition is the least perimeter way to partition 24 equal area regions on M.*

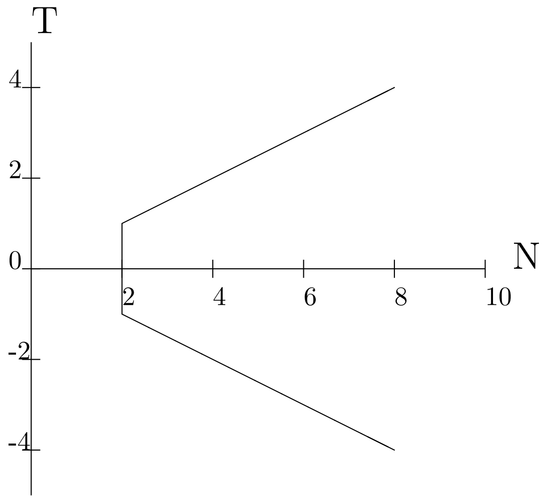

Figure 1: Possible values of N and T.

Proof. Let $\{R_i\}$ be a partition of M into 24 equal area regions. We may assume that each region is the finite union of curvilinear polygons $R_i = union P_i$ with vertices meeting in threes [4]. Each P_i satisfies the area requirement of Lemma 2.2 $\frac{2\pi}{\sqrt{3}N(P_i)^2} < Area(P_i) < \frac{\pi}{3}$.

Each edge e_i of P_i occurs with opposite orientation as an edge e_j on an adjacent region P_j, therefore the area $x(e_i)$ is identical in magnitude and opposite in sign to $x(e_j)$. Summing over all the edges of the partition gives $\sum T(P_i) = 0$.

Applying the heptagonal isoperimetric inequality, the total perimeter of the partition is bounded by

$$2Perimeter = \sum L(P_i) \geq \frac{3}{\pi} P_7 \sum A(P_i) - \sum T(P_i) \frac{P_7}{2} - .112 \sum (N(P_i) - 7)$$

$$\geq 24 P_7.$$

Thus the perimeter is at least $12P_7$, the length of the perimeter of the regular hexagonal partition. □

3 Proof of the Heptagonal Isoperimetric Inequality

To prove Lemma 2.2 it is sufficient to demonstrate that it holds for all possible values of N, T, and all $A > \alpha(N)$. Let $F(N, T, A) = \frac{3}{\pi} P_7 A - T \frac{P_7}{2} - .112(N - 7)$ and $\Delta P = L(P) - F$. Then it suffices to show that $\Delta P \geq 0$.

By the definition of N and T we know immediately that $N \geq 2|T|$ and $N \geq 2$. (See Figure 1.)

First we establish some useful lower bounds for the perimeter P of a curvilinear polygon as function of the number of sides N and the enclosed area A. Next we demonstrate that the theorem is true by proving it for digons, when $N = 2$. Then we prove the remaining cases where $N \geq 3$.

3.1 Lower Bounds

The following bounds hold in the hyberbolic plane of constant curvature -1.

$\mathbf{L_N}$: The perimeter is at least as big as L_N, the perimeter of a regular N-gon.

$$P \geq L_N = 2N cosh^{-1}\left(\frac{cos\frac{\pi}{N}}{sin\left(\frac{\pi(N-2)-A}{2N}\right)}\right)$$

$\mathbf{L_+}$: The perimeter is at least as big as the circumference of a circle enclosing the same area.

$$P \geq L_+ = \sqrt{A(4\pi + A)}$$

$\mathbf{L_-}$: The perimeter is at least as great as the circumference of a circle having the area of the modified polygon obtained by reflecting each inward curving edge outward across the chord.

Let J be the set of indices of the edges that curve inward, that is, the set of edges with $x(e_j) \leq 0$ and $X_J = \sum_{j \in J} x(e_j)$.

$$P \geq L_- = \sqrt{(A - 2X_J)(4\pi + A - 2X_J)}$$

$\mathbf{L_D}$: According to Dido a semicircle enclosing a given area has a smaller perimeter than any other arc of a circle enclosing the same area. Applying this to all the edges and letting $X_D = \sum |x(e)|$ we have another lower bound.

$$P \geq L_D = X_D\sqrt{4\pi + 1}$$

3.2 Digons (N=2)

Suppose that $N = 2$ so P is a digon. Then $A(P) = x(e_1) + x(e_2)$.

If $T \geq .425$, then $\Delta P = L(P) - F(2, .425, A)$, since F is a decreasing function of T. Using the bound L_+ we have

$$\Delta P \geq \sqrt{A(4\pi + A)} - \frac{3}{\pi}P_7 A + .425\frac{P_7}{2} + .112(2 - 7)$$

$$> \sqrt{A(4\pi + A)} - \frac{3}{\pi}P_7 A + .282308 > 0$$

if $A \in [.85, \frac{\pi}{3}]$.

Suppose $T < .425$. Since $A = x_1 + x_2 > .85$, $x_{max} > \frac{1}{2}$ and

$$T = x_{min} + \frac{1}{2} < .425$$

$$x_{min} < -.075.$$

Apply the lower bound L_-, in this case with the single negative area x_{min}.

$$\Delta P = L(P) - F(2,T,A) \geq \sqrt{(A - 2x_{min})(4\pi + A - 2x_{min})} - \frac{3}{\pi} P_7 A + .425 \frac{P_7}{2} - .56 > 0$$

for $A \in [.85, \frac{\pi}{3}]$, $x_{min} \in [-.5, .075]$.

3.3 N ≥ 3

For these cases, the two bounds L_N and L_- cover all of the NT-plane except for the following intervals.

$$N = 4, T \in [-.344, -.095]$$
$$N = 5, T \in [-.257, .004]$$
$$N = 6, T \in [-.179, .018]$$
$$N = 7, T \in [-.107, 0]$$
$$N = 8, T \in [-.041, -.033]$$

These intervals can be verified individually.

References

[1] A. L. Edmonds, J. H. Ewing, R. S. Kulkarni, *Regular Tessellations of Surfaces and (p,q,2)-triangle Groups*, The Annals of Mathematics, Second Series, Volume 116, Issue 1 (July, 1982), 113-132.

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[4] F. Morgan, *Soap Bubbles in* $\Re^2$ *and in Surfaces*, Pacific J. Math, 165 (1994), no. 2, 347-361.

[5] F. Morgan, *Geometric Measure Theory*, Academic Press, San Diego, 3rd Ed., 2000.