On Generalizing the Honeycomb Theorem to Compact Hyperbolic Manifolds and the Sphere

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Abstract. We provide a possible alternate proof to the Honeycomb Conjecture in the plane. We generalize the proof of the hexagonal isoperimetric inequality to $\mathbf{S}^2$ and $\mathbf{H}^2$ under certain conditions and deduce that a nice valence-three tiling by regular polygons of $\mathbf{S}^2$ or of a compact hyperbolic manifold is perimeter minimizing.

Contents

1. Introduction

The Honeycomb Theorem says that regular hexagons provide the least perimeter way to partition the plane (or appropriate torus) into unit areas. It was mentioned in Varro's *On Agriculture* in 36 BC. It remained one of the oldest open problems in geometry until Hales announced his proof in 1999 ([3], see [6] Ch. 15).

We propose a simpler, more geometric version of the proof and generalizations to compact hyperbolic manifolds and the sphere. For the sphere, it is conjectured that the tetrahedral and cubic arrangements provide perimeter-minimizing partitions of the sphere into four and six equal areas, respectively. Hales [4] proved that the dodecahedral arrangement provides a perimeter-minimizing partition of the sphere into twelve equal areas. For a compact hyperbolic manifold, Cox [1] and Šešum [7] conjecture that any partition into regular N-gons minimizes perimeter for given areas. Our proposition 6.1 proves this Honeycomb conjecture on compact hyperbolic manifolds, assuming Conjecture 5.1, a generalization of Hales's Hexagonal Isoperimetric Inequality 4.3.

Hales's Hexagonal Isoperimetric Inequality says that regular hexagons solve a local isoperimetric problem among curvilinear polygons with penalties for bulging and more than six sides. We show that the generalized Polygonal Isoperimetric Inequality for compact hyperbolic manifolds, Conjecture 5.1, *fails* for $N > 66$, although it seems true for $7 \leq N \leq 12$.

There are two main steps in the proposed form of Conjecture 5.1. First, we prove a generalization (Prop. 2.8) of Hales's Chordal Isoperimetric Inequality ([3], 6.1). The second step is the unproven convexity of a certain geometric function (Conjecture 4.1).

We will also discuss a failed method for simplifying the proof of the Honeycomb Theorem, which attempted to use regularity results to imply the Theorem. The general theory says that, given a smooth compact Riemannian surface M and positive areas A_i summing to the total area of M into regions of area A_i, there exists a least perimeter partition of M, by constant-curvature curves meeting in threes at 120 degrees. At each vertex, the sum of the curvature is zero ([5], Thm. 2.3 and Cor. 3.3).

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2. Hyperbolic and Spherical Isoperimetric Inequalities

This section records classical isoperimetric inequalities for $\mathbf{H}^2$. We provide a new, simple proof of one isoperimetric inequality for n-gons in Proposition 2.5.

Proposition 2.8 and Corollary 2.9 generalize Hales's key Chordal Isoperimetric Inequality ([3], Prop. 6.1) to $\mathbf{S}^2$ and $\mathbf{H}^2$. This inequality plays a key role in the proof of the Polygonal Isoperimetric Inequality, Conjecture 5.1, which implies the Honeycomb Conjecture on compact hyperbolic manifolds, Proposition 6.1.

Definition 2.1. For a curvilinear polygon P, consider the perimeter $L(P)$, enclosed area A, and signed area between edge and chord x_i, where $\Sigma x_i = X$.

Proposition 2.2 (L_+). *The perimeter of a curvilinear polygon is at least as big as the circumference of a circle enclosing the same area:*

$$L(P) \geq L_+ = \sqrt{A(4\pi + A)}.$$

Proposition 2.3 (L_-). *The perimeter of a curvilinear polygon is at least as great as the circumference of a circle having the area of the modified polygon obtained by reflecting each inward bulging edge outward across the chord. Let J be the set of indices of the edges that bulge inward, that is, the set of edges with $x_j \leq 0$, and let $X_J = \Sigma_J x_j$.*

$$L(P) \geq L_- = \sqrt{(A - 2X_J)(4\pi + A - 2X_J)}.$$

Proposition 2.4 (L_D). *A semicircle enclosing a given area has a smaller perimeter than any other constant-curvature arc enclosing the same area. Applying this inequality to all the edges and letting $X_D = \Sigma|x_j(e)|$ yields*

$$L(P) \geq L_D = X_D\sqrt{4\pi + 1}.$$

Proposition 2.5 (L_N). *The perimeter of a curvilinear polygon with $X \leq 0$ is at least as great as L_N, the perimeter of a regular n-gon.*

$$L(P) \geq L_N = 2N cosh^{-1}\left(\frac{cos\left(\frac{\pi}{N}\right)}{sin\left(\frac{\pi(N-2)-A}{2N}\right)}\right).$$

Proof. We show that for fixed perimeter, the regular n-gon maximizes area. Given a curvilinear polygon P with bulging in edges, area is increased and perimeter decreased by replacing the edges with geodesics; it therefore suffices to consider n-gons (with geodesic edges).

Consider a triangle with fixed base length and side lengths x and y. Using Heron's formula generalized to the hyperbolic plane, we find that

$$tan^2\frac{A}{2} = \frac{a - cosh^2(x) - cosh^2(y)}{b + cosh(x) + cosh(y)}$$

for some constants a, b. Fixing $x+y$, which fixes perimeter, area is maximized when $x = y$, which maximizes the numerator and minimizes the denominator. It follows that equilateral n-gons are the most efficient n-gons.

Next we show that n-gons inscribed in circles are more efficient than general n-gons. Suppose that the vertices of P do not all lie on a hyperbolic circle. Move the vertices now so that they are on a hyperbolic circle, and bulge out each side such that we have a circle. We can move the vertices back to their original positions and put the bulges on the corresponding sides. The bulging adds the same area in either case, but if the case where the vertices are not on a circle encloses more area, then this bulging polygon would enclose more area given a perimeter than the circle, a contradiction. Therefore, the regular n-gon is the most efficient chordal n-gon.

□

Definition 2.6. Let $arc(x)$ be the length of the constant curvature arc enclosing area x over a chord of length one.

We need the following lemma before we can generalize Hales's chordal isoperimetric inequality.

Lemma 2.7. *In* $\mathbf{S^2}$, $\mathbf{R^2}$, *and* $\mathbf{H^2}$, *for* $\lambda > 1$, $arc(\lambda x) < \lambda\ arc(x)$.

Proof. We begin by showing Lemma 2.7 is true for all integers k. Consider a constant curvature arc enclosing area x over a chord of length 1, with length $arc(x)$. Two identical such arcs together enclose area $2x$ with chord length $2arc(x)$ over a chord of length 2. Since a single constant curvature arc encloses area more efficiently, a single arc of length $2arc(x)$ over a chord of length 2 will enclose more than $2x$ area. So $2arc(x) > arc(2x)$.

This argument holds for scaling by any k: $arc(kx) < karc(x)$.

Similarly, to prove Lemma 2.7 for any rational q, we show by contradiction that $arc(x/k) < \frac{arc(x)}{k}$. If an arc of length $arc(x)/2$ encloses area greater than or equal to $x/2$, then an arc of length $arc(x)$ would enclose more than x area, contradicting the statement above.

Again, this argument holds for any integer k: $arc(x/k) < \frac{arc(x)}{k}$. Thus Lemma 2.7 is true for all rationals q, and, since the rationals are a dense subset of the reals, we can scale continuously by any real λ.

□

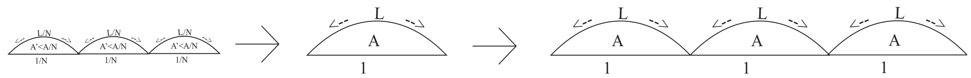

FIGURE 1. Graphically scaling up and down.

The following Chordal Isoperimetric Inequalities 2.8 and 2.9 play a key role in the proof of the Polygonal Isoperimetric Inequality 5.1. The proof follows the Euclidean proof, ([[3], Appendix 1], [[6], Prop. 15.5]).

Proposition 2.8 (Chordal Isoperimetric Inequality). *Consider an immersed curvilinear polygon P in $\mathbf{H^2}$ as in figure 2 of perimeter L, net excess area X over the chordal polygon, where $|X| \leq \pi/8$, with each chord length at most 1. Let L_0 be the perimeter of the chordal polygon. Then*

$$L \geq L_0 arc(\frac{|X|}{L_0}).$$

The function $L_0 arc(|X|/L_0)$ is increasing with respect to L_0, for fixed X.

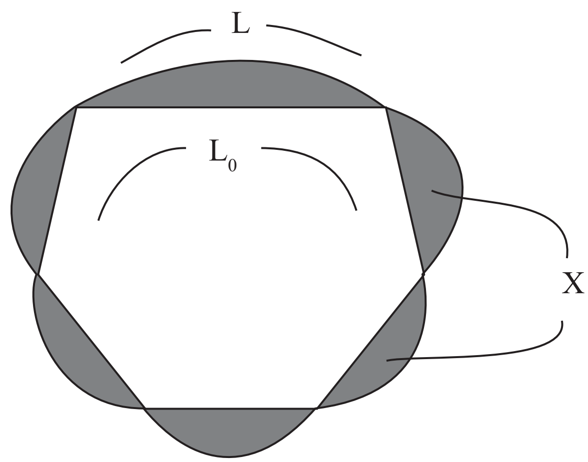

FIGURE 2. An immersed curvilinear polygon with perimeter L, excess area X over the chordal polygon.

Proof. We may assume each edge of the polygon is a constant curvature arc. The derivative of $arc(x)$ is the curvature of the arc, so it is convex up to the semicircle where $|X| = 2\pi sinh^2(1/4) \approx 0.401 = X_0$. Now we show that the arcs are at most semicircles.

We may assume that the excess area of each side x_i has the same sign.

We now consider the less constrained problem of enclosing area X by arcs above chords on the x-axis of length at most 1 and total chord length L_0. The minimizer must consist of constant curvature arcs all of the same curvature, since curvature is the rate of change of length with respect to area.

Given two bulges of area A and B with $A \geq B$, subtending angles of a and b, respectively, as in figure 2, we claim that the most efficient way of enclosing area $A + B$ is with all the area under one arc.

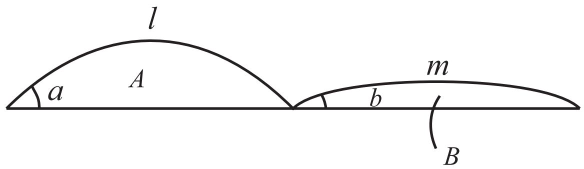

FIGURE 3. Two chords of length 1 surmounted by circular arcs of length l and m, enclosing areas A and B, respectively.

Since $a > b$, more area would be enclosed by equal perimeter by taking a section off of the smaller bulge and adding it to the larger bulge (see figure 2). The perimeter of each bulge could then be smoothed back out into circular arcs and shortened until the enclosed area is again $A+B$. Because the derivative with respect to moving lower arcs to higher arcs is always positive, the area is maximized when the area of one buldge is zero.

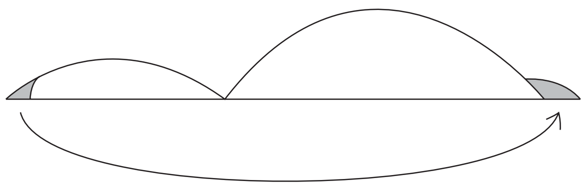

FIGURE 4. It pays to make the larger arc still larger.

Hence if $L_0 \leq 1$, the minimizer is one arc over a chord of length L_0, and its length is greater than

$$L_0 arc(\frac{X}{L_0}),$$

by Lemma 2.7.

If $L_0 > 1$, then the minimizer is $m \geq 1$ arcs over chords of length $L_i = 1$ and one arc over a chord of length $0 \leq L_{m+1} < 1$. The excess area is limited to a semicircle because $X \leq \pi/8$. Then

$$L < \Sigma L_i arc(\frac{X}{L_i}) \leq L_0 arc(\frac{X}{L_0}),$$

by convexity, as desired.

Finally, we show that $L_0 arc(|X|/L_0)$ is increasing, or equivalently that $arc(x)/x$ is decreasing, or equivalently that $arc'(x) < arc(x)/x$. Since $arc(x)$ is convex for $0 \leq x < X_0$ and concave for $X_0 < x$, it suffices to check this at $X = X_0$ (the semicircle of radius 1/2), where indeed the curvature

$$arc'(X_0) = 2 < (2\pi sinh(1/2))/(4\pi sinh^2(r/2)) \approx 4.08.$$

□

Corollary 2.9. *The chordal isoperimetric inequality holds in* $\mathbf{R^2}$ *and* $\mathbf{S^2}$*.*

Proof. Consider a polygon P_H in the hyperbolic plane with the same area, perimeter and net excess area as P. Let P_H have chordal perimeter L_{0_H} and arc function $arc_H(|X|/L_{0_H})$. Then $L_0 < L_{0_H}$ and $arc(B) < arc_H(B)$, for any $0 \leq B \leq \pi/8$. By Proposition 2.8, $L \geq L_{0_H} arc_H(X/L_{0_H})$, and $L_{0_H} arc_H(|X|/L_{0_H})$ is increasing in L_0, so

$$L \geq L_{0_H} arc_H(\frac{X}{L_{0_H}}) \geq L_0 arc(\frac{X}{L_{0_E}}).$$

The spherical case follows similarly from the hyperbolic or Euclidean case. □

3. An Oversimplified Approach

This section gives an oversimplified approach to proving the Honeycomb Theorem. In a perimeter-minimizing partition, each component polygon has an average of six sides by Euler. It has net bulge of zero, since a positive bulge on one polygon is equally balanced by a negative bulge on an adjacent polygon. Were these facts true for each polygon individually, the Honeycomb Conjecture would follow immediately.

Indeed, suppose that the perimeter-minimizing partition consisted of hexagons with net bulge zero. By equilibrium, one can assign a pressure to each region such that the curvature of an edge is the difference of the pressures of the regions it separates.

Take a highest pressure region in the partition, which must be convex. Since the net bulge must be zero, this component is a hexagon. Now consider a next highest pressure region. By the same reasoning, this must be a hexagon. Similarly, every region is a regular hexagon, and by Proposition 2.5 regular hexagons are the best hexagons.

Unfortunately this idea breaks down from the start. For n-gons with $n \neq 6$, we have to allow net bulging. Worse, the ideal circle, more efficient than the regular n-gon, may be approximated by n circular arcs meeting at 120 degrees, as in Figure 5.

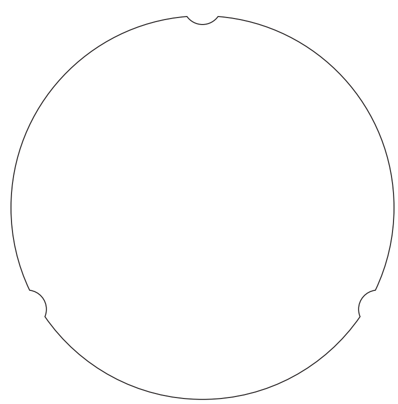

Figure 5. A very efficient hexagon with constant curvature arcs meeting at 120-degrees.

4. The Hexagonal Isoperimetric Inequality

Hales's proof of the Honeycomb Theorem utilizes a Hexagonal Isoperimetric Inequality, Theorem 4.3, which linearly represents the perimeter gains that a polygon makes by approaching a circle either by increasing the number of sides or by bulging out (increasing n or X). Thus when we average across all of the polygons in a compact region using the Hexagonal Isoperimetric Inequality, we can apply the knowledge we already know about the whole minimal partition and by extension about the average polygon in the partition to find the perimeter of this average polygon. We get the perimeter of regular hexagon with unit area.

In order to prove the Hexagonal Isoperimetric Inequality, Hales finds lower bounds on perimeter $L(P)$ which are greater than the Hexagonal Estimate $F(P)$ for each P. The proof of the Hexagonal Isoperimetric Inequality consists of finding some intermediate bound L_b for each point (A, N, X) in the parameter space of area, number of sides, and truncated bulging area such that $L_b > F(A, N, X)$. The proof becomes a long, arduous case analysis using five seperate intermediate lower bounds.

We propose to simplify this proof by generalizing the case $N = 6$ and making greater use of the principles which make the Hexagonal Isoperimetric Inequality true in this case. We will also assume that $A = 1$ since generalizing from $A = 1$ is relatively easy and involves only the fact that if we scale the area of polygons down, perimeter scales slower. We will also avoid Hales's truncation function on the bulging area of the polygon.

In this case, the hexagonal estimate reduces to $F(X) = 2\sqrt[4]{12} - X\sqrt[4]{12}$. Since $F(X)$ is a line and $F(X) = L(P)$ at $X = 0$ and P regular, the bound is the tangent line to $P_6(X)$, the perimeter of a regular hexagon with unit area in terms of X. Proof of the bound would follow for all P regular simply by the convexity of $P_6(X)$. To finish the $N = 6$ case, one could show that an irregular polygon with any X does not have perimeter so much smaller than that of the regular one that it violates the inequality. This proof can be achieved by showing that $F(X)$ lies under not just $P_6(X)$ but also a lower bound estimate for perimeter. The chordal isoperimetric inequality is ideal because the bound is tight at $X = 0, P$ regular and hence tangent to $F(X)$ and $P_6(X)$ at $X = 0$. Cases outsides the relevent range of chordal isoperimetric inequality are not tight and therefore do not require as much subtlety.

We generalize this approach to different values of N. Start by defining $P_n(X)$ to be the perimeter of a regular curvilinear n-gon in the Euclidean plane with congruent circular arc edges, unit area, and excess area X over the n-gon with straight edges. Define $b_n = -P'_n(0)$.

For any curvilinear n-gon P, define

$$F_n(P) = A(P)P_n(0) - X(P)b_n. \tag{1}$$

For regular curvilinear n-gons P, F_n is linear and tangent to $P_n(X)$ at $X = 0$. Note that F_n reduces to $F(X)$ in the case $n = 6$.

Our proposed simplification starts by showing that the bound given by the chordal isoperimetric inequality is convex.

Conjecture 4.1. *Define L_0 to be the perimeter of a regular polygon with area $I = A - X$. The function $L_0 arc\left(\frac{|X|}{L_0}\right)$ is convex as a function of X when $-\pi/8 < X < \pi/8$ and $A = 1$.*

The proof of this conjecture is incomplete, but we provide supporting graphical evidence from a mathematica plot of $L_0 arc\left(\frac{|X|}{L_0}\right)$ with respect to X. Even if convexity fails, as it appears to towards the end of the range, we could still show that the tangent line at 0 stays underneath.

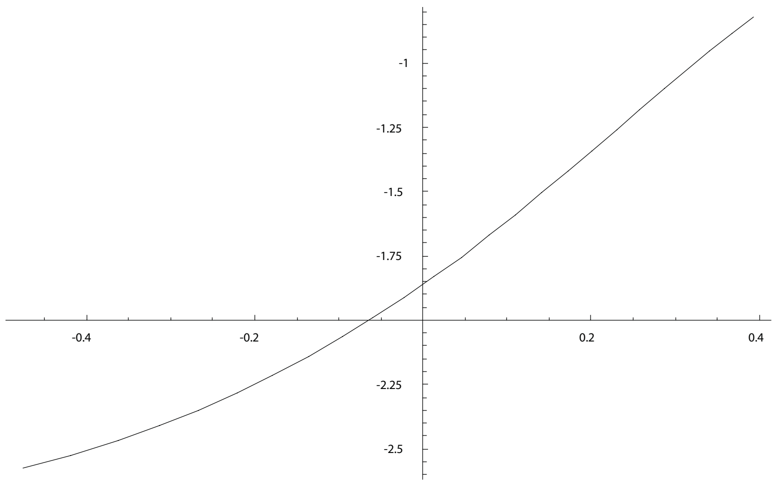

FIGURE 6. A plot showing the derivitive of $L_0 arc\left(\frac{X}{L_0}\right)$. Note that the axes are negative, but that the function is always increasing, indicating convexity.

Proposition 4.2. *Assuming Conjecture 4.1, if P is a curvilinear n-gon with $A(P) \leq 1$, then the perimeter $L(P) \geq F_n(P)$.*

Showing that P_n is convex would be sufficient to prove Proposition 4.2 for P regular with unit area. Likewise, a simple scaling argument suffices to expand Proposition 4.2 to P regular with area less than one. However, the case of P irregular is harder to handle and requires Conjecture 4.1.

Assuming Conjecture 4.1, we can almost prove Proposition 4.2 using the method we outline for $F(X)$ when $n = 6$.

Proof. By the Chordal Isoperimetric Inequality 2.8, we know that $L(P) \geq F(A(P), X(P))$. Hence $P_n(X) \geq F(1, X)$ with equality at $X = 0$. Thus, $F(1, X)$ and $P_n(X)$ are tangent at $X = 0$. So $F_n(P)$ is the tangent line to $F(1, X(P))$ as well. Convexity demonstrates that this tangent line is below the curve. Putting it all together, for a curvilinear n-gon P with $-\pi/8 < X(P) < \pi/8$ and $A(P) = 1$, we know $L(P) \geq F(1, X(P)) \geq F_n(P)$.

Now we must deal with the case when $|X(P)| \geq \pi/8$ and $A(P) = 1$. The case $X(P) \geq \pi/8$ turns out to be relatively easy since, for the most part, increasing positive bulging sum $X(P)$ increases the perimeter of polygons and the lower bound

F_n is actually decreasing. When $F_n < 2\sqrt{A\pi}$, for example, the F_n bound is proven by the isoperimetric inequality which says that $L(P) > 2\sqrt{A(P)\pi}$ the perimeter of a circle of equivilent area. Since we are in the case $A = 1$, it suffices to show F_n is true between $\pi/8$ and some M_n such that $F_n(M_n) = 2\sqrt{\pi}$. When $X(P) > M_n$, $F_n < 2\sqrt{\pi}$. We compute

$$M_n = \frac{P_n(0) - 2\sqrt{\pi}}{b_n} = 2\left(1 - \frac{\sqrt{\pi}}{\sqrt{n \tan\frac{\pi}{n}}}\right) \tag{2}$$

by using the formula that $P_n(0) = \sqrt{4nA\tan\pi/n}$ and by computing the derivative b_n using the method in Proposition 5.3. We get that $M_3 = 2(1 - \sqrt{\pi}/3^{3/4}) \approx 0.44488$ and $M_4 = 2 - \sqrt{\pi} \approx 0.227546$. Since $\pi/8 \approx 0.392699$, the case where X is between $\pi/8$ and M_n does not exist for $n = 4$. So the case $X(P) \geq \pi/8$ is finished for $n = 4$. In fact, we claim that M_n is a decreasing function of n and thus the case where $X(P) \geq \pi/8$ is finished for $n \geq 4$. To show that M_n is a decreasing function of n, we calculate

$$\frac{d}{dn}M_n = \frac{\pi\left(-\frac{\pi\sec^2\left(\frac{\pi}{n}\right)}{n} + \tan\left(\frac{\pi}{n}\right)\right)}{n\tan^{\frac{3}{2}}\left(\frac{\pi}{n}\right)}. \tag{3}$$

To show $\frac{d}{dn}M_n$ is less than zero, we need only show that

$$\tan\left(\frac{\pi}{n}\right) < \frac{\pi}{n}\sec^2\left(\frac{\pi}{n}\right). \tag{4}$$

Dividing by $\sec^2(\pi/n)$ and then using trigonometric identities shows that this condition is equivalent to $\sin(\pi/n)\cos(\pi/n) < \pi/n$. Another identity gives $(1/2)\sin(\pi/(2n)) < \pi/n$. We are only considering $n > 2$, so we can overestimate sin by considering a line through the origin with unit slope. This estimation gives the harder condition, $(1/2(\pi/(2n)) < \pi/n$. Since $1/4 < 1$, we are done; M_n is decreasing.

Hence the last step of the $X(P) \geq \pi/8$ case is $n = 3$ and $\pi/3 < X < 2 - \sqrt{\pi}$, which has yet to be concluded. We now look at the case $X(P) < -\pi/8$. We will assume that each edge of P bulges inward since if one edge bulged outward, we could reduce perimeter by changing the outward bulging edge to a chord and reducing the curvature of an inward bulging edge to keep $X(P)$ and $A(P)$ constant. We will use here the bound that Hales calls L_-. This bound is formed by noting that a polygon has greater perimeter than the perimeter of a circle whose area is equal to the area of the original polygon with its inward bulging edges flipped out, that is, $L(P) \geq L_-(P) = 2\sqrt{(A - 2X_I)\pi}$ where X_I is the sum of the areas between the inward bulding edges and their respective chords.

For this bound to imply the inequality $L(P) > F_n(P)$, we need to show that $L_-(P) > F_n(P)$. Since $T(P) < -\pi/8$ and all the edges are inward bulging, this statement is equivalent to $2\sqrt{(1 - 2X)\pi} \geq (2 - X)\sqrt{n\tan(\pi/n)}$. Both sides are positive, so we can square each side and rearrange terms giving,

$$\frac{1 - 2X}{(2 - X)^2} \geq \frac{n\tan\left(\frac{\pi}{n}\right)}{4\pi}. \tag{5}$$

Differentiating the left-hand side with respect to X gives

$$\frac{2(X-1)}{(2-X)^3}, \tag{6}$$

which is negative on the interval $-1 \geq X \leq -\pi/8$. Hence, we can find the lowest value of the left-hand side by evaluating at $-\pi/8$ which is approximately 0.311859. So the inequality holds when $0.311 > n\tan(\pi/n)/(4\pi) = Y_n$. We know Y_n is decreasing because we proved M_n was decreasing. Since $Y_4 \approx 0.31831$ and $Y_5 \approx 0.289082$, we know we are done for $n \geq 5$. Going back and solving in the cases where $n = 3, 4$ for the value of X where the inequality stops working gives no real value for $n = 3$ and the value -0.47464 for $n = 4$. Thus the cases that remain for $X(P) < -\pi/8$ are $n = 3, -1 < X < -\pi/8$ and $n = 4 - 0.475 < X < -\pi/8$.

After handling these cases and the case $n = 3$ and $\pi/3 < X < 2 - \sqrt{\pi}$, one could conclude $L(P) \geq F_n(P)$ when $A(P) = 1$.

Now consider an n-gon P_0 for which $A(P_0) < 1$. Let $\epsilon = A(P_0)$ the scaling factor. Then we can scale up the polygon P_0 uniformly by a factor of $1/\epsilon$ to get a new polygon P_1 with $\epsilon A(P_1) = A(P_0)$ and $\epsilon X(P_1) = X(P_0)$ but $\sqrt{\epsilon}L(P_1) = L(P_0)$, which implies $\epsilon F_n(P_1) = F_n(P_0)$. Since the proposition holds for P_1, we have $L(P_1) \geq F_n(P_1)$. Substituting gives $L(P_0) \geq \sqrt{\epsilon}F_n(P_0)$. Since $1 > \sqrt{\epsilon} > 0$, this gives $L(P_0) \geq F_n(P_0)$. So the inequality holds for P_0. □

Now we have constructed bounds $F_n(P)$ which apply only to n-gons with matching n. These bounds are easier to prove than the general bound $F(P)$ since they have the "correct" slope. Now we will use these bounds to prove the bound $F(P)$, thinking of the $(N(P) - 6)0.0505$ term as a correction for the fact that we are not using the "correct" slope. Note that in our new notation, $F(P)$ now has the more concise form,

$$F(P) = P_6(0)A(P) - X(P)b_6 - (N(P) - 6)c. \tag{7}$$

where $c = 0.0505$.

Now, the proof of Theorem 4.3 is much simplified. Only a couple of cases should be left unsolved.

Remark In the Hexagonal Isoperimetric Inequality, Hales truncates the bulging area on each edge x_i to have at most absolute value 1/2. For now we deal only with the case $|x_i| < 1/2$. The other case is not as difficult; see [3].

Theorem 4.3 (The Hexagonal Isoperimetric Inequality (Hales [3]; See [6], Prop 15.4))**.** *Let P be a curvilinear embedded polygon with n edges and area $A(P)$, where $X(P)$ is defined as the sum of the signed areas between the arcs of the polygon and the chords between the vertices (perhaps truncated in general). Then*

$$L(P) \geq F(P) = A(P)2\sqrt[4]{12} - X(P)\sqrt[4]{12} - (N(P) - 6)0.0505. \tag{8}$$

Proof of Theorem 4.3, assuming Conjecture 4.1. We start with the case $A = 1$. Note that the case $A < 1$ and $n \geq 6$ follows from the $A = 1$ case. Assume the theorem holds for all P where $A(P) = 1$. Let P_0 be a curvilinear polygon such that $N(P_0) \geq 6$ and $A(P_0) \leq 1$. Then we can scale the area of P_0 up by a factor of $\epsilon > 1$ to get a polygon P_1 such that $A(P_1) = \epsilon A(P_0) = 1$. Now, since the Theorem holds for P_1,

$$L(P_1) + k \geq A(P_1)P_6(0) - X(P_1)b_6 \tag{9}$$

where $k = (N(P_1) - 6)c$. Since $N(P_1) \geq 6$, $k \geq 0$. Multiplying the right-hand side by $1/\sqrt{\epsilon}$ preserves the inequality since $1/\sqrt{\epsilon} < 1$, so

$$L(P_1) + k \geq \frac{1}{\sqrt{\epsilon}}(A(P_1)P_6(0) - X(P_1)b_6), \tag{10}$$

and mulpilying by $1/\sqrt{\epsilon}$ gives

$$\frac{1}{\sqrt{\epsilon}}(L(P_1) + k) \geq \frac{1}{\epsilon}(A(P_1)P_6(0) - X(P_1)b_6). \tag{11}$$

Substituting, we get,

$$L(P_0) + \frac{k}{\sqrt{\epsilon}} \geq A(P_0)P_6(0) - X(P_0)b_6, \tag{12}$$

and hence,

$$L(P_0) + k \geq A(P_0)P_6(0) - X(P_0)b_6, \tag{13}$$

because $k \geq \frac{k}{\sqrt{\epsilon}}$. The case of $2 \leq N(P) \leq 5$ may be more complex.

To do the case $A = 1$, we will employ our $F_n(P)$ bounds. Fix $N(P) = n$ and $A(P) = 1$. Now, we are left with one parameter $X = X(P)$. We need only consider $|X| \leq 1$, since $|X| < A$. Our hope is that for the relevant range of X, $F_n(X) > F(X)$. Our first step is to find the intersection of the lines $F_n(X)$ and $F(X)$. Setting equal gives,

$$P_6(0) - Xb_6 - c(n-6) = P_n(0) - Xb_n. \tag{14}$$

Solving for X, we find

$$(P_6(0) - P_n(0) - c(n-6))/(b_6 - b_n) = X. \tag{15}$$

Call this point of intersection X_n. Computation of X_n shows that X_n lies completely outside the relevant range of X for $n = 3, 4, 10, 11, and 12$. For $n > 12$, computation suggests that X_n is moving further and further from the relevant range of X. Thus, the only cases where the inequality would have to be proved independently would be $5 \geq n \geq 9$ when $F(X) \geq F_n(X)$. In each of these cases, $F_n(X)$ proves the inequality for at least half of the range of X.

□

5. The Polygonal Isoperimetric Inequality in $\mathbf{H}^2$

We now try to generalize the Honeycomb Conjecture to compact hyperbolic manifolds. Given a partition of a compact manifold by regular polygons meeting in threes, we conjecture that this partition is minimal. Unlike the Euclidean plane, there exists a hyperbolic manifold which can be partitioned into regular polygons meeting in threes for each $N \geq 7$ (See Edmonds [2], Main Theorem). So in place of a hexagonal isoperimetric inequality, we conjecture a polygonal isoperimetric inequality for $7 \leq N \leq 66$ from which we will prove that a given partition by regular n-gons is minimal. The existence of an upper bound on the N for which such an inequality would hold was surprising but seems to be justified by the evidence presented at the end of this section.

Conjecture 5.1 (The Polygonal Isoperimetric Inequality in $\mathbf{H}^2$)**.** *Fix $7 \leq N \leq 12$. Let P be a curvilinear embedded polygon with n edges and area $A(P) \leq \frac{N-6}{3}\pi = A_N$*

the area of a regular N-gon with 120-angles, where X is defined as above. Then, for every such N, there exists a pair of real numbers (b,c) such that

$$L(P) \geq \frac{A(P)}{A_N} P_N - bX(P) - c(n-N),$$

where P_N is the perimeter of a regular N-gon of area $A_N = \frac{N-6}{3}\pi$.

Remark We use the upper bound 12 on N because we were unable to prove Proposition 5.2 for large N, though the upper bound could be larger. However, our counterexample demonstrates that the upper bound on N is less than 66.

We believe that the proof will follow from a family of bounds in a similiar manner to Hales's Euclidean proof. We will generalize an isoperimetric estimate for chordal polygons to $\mathbf{H}^2$.

We will also find values for b and c and provide a counterexample to Conjecture 5.1 for some cases.

We begin with the following simple case:

Proposition 5.2. *Conjecture 5.1 holds for $N = 7, 8, \ldots, 12$, $n = N, X = 0$.*

Proof. We begin with the formula supplied by Sesum [7],

$$L_N = 2N cosh^{-1}\left(\frac{cos\left(\frac{\pi}{N}\right)}{sin(\frac{\pi(N-2)-A}{2N})}\right). \tag{16}$$

Further, we make the substitutions

$$\begin{aligned} L &= \frac{L_N}{2N} \\ a &= \frac{\pi}{N} \\ \hat{A} &= \frac{A}{2N}. \end{aligned}$$

Then (1) reduces to

$$cosh(L)cos(a+\hat{A}) = cos(a),$$

so differentiating implicitly,

$$\begin{aligned} L' &= coth(L)tan(a+\hat{A}) \\ L'' &= cos(a)sec^3(a+\hat{A})(1 - \frac{sin^2(a+\hat{A})}{sinh^2(L)}), \end{aligned}$$

but $cos(a)$ is a positive constant and $sec(a+\hat{A}) > 0$, so $sec^3(a+\hat{A}) > 0$. So it suffices to show that

$$1 - \frac{sin^2(a+\hat{A})}{sinh^2(L)} < 0,$$

or just that

$$sin^2(a+\hat{A}) > sinh^2(L),$$

i.e.,

$$cosh^2(L) + cos^2(a+\hat{A}) < 2.$$

To prove this inequality, we evaluate at the endpoints and critical points. The endpoints are given by $\hat{A} = 0$, $\hat{A} = \pi/6$. The critical points can be solved for by differentiating the left-hand side of the above inequality. There is only one in the relevant range. It is $cos^2(a+\hat{A}) = \sqrt{1/cos^2(a)}$.

Evaluation of these three points for $N = 7, \dots, 12$ gives values less than two. Hence, the inequality holds over the relevant range. So $L'' < 0$. By the chain rule, this implies $L''_N < 0$. Thus $\Delta L = L_N - 3AP_N/\pi$ is concave down. Evaluation at the endpoints shows $\Delta L \geq 0$ on the whole interval. This implies the isoperimetric inequality in this special case.

□

Lemma 5.3. *For $N = 7$, the value of b is approximately* 2.09338*.*

Proof. Let $L(A, X)$ be the perimeter of a curvilinear 7-gon with area A and bulge X. Then $A = a + X$, where a is the area of a 7-gon formed by connecting the vertices of the given 7-gon with geodesics. Then for $a = a_0$

$$\frac{\partial L}{\partial X}|_{a=a_0} = \frac{\partial L}{\partial A} + \frac{\partial L}{\partial X} \Rightarrow \frac{\partial L}{\partial A} - b = 0 \tag{17}$$

So we solve for $\frac{\partial L}{\partial A}$ as follows:

$$\frac{\partial}{\partial A} 14 arccosh(\frac{cos(\frac{\pi}{7})}{sin(\frac{5\pi - A}{14})})|_{A=\pi/3} \approx 2.09338$$

□

Below is the surprising counterexample to Proposition 5.1 for $N > 66$.

Proposition 5.4. *The Polygonal Isoperimetric Inequality (Proposition 5.1) does not hold for large N, large A.*

Proof. Consider the case where $X = 0$, $n = N$. We will compare the Polygonal Isoperimetric Inequality to the perimeter of a regular N-gon, which is given to us as the lower bound L_N, and is explicitly:

$$L(N, A) = 2cosh^{-1}\left(\frac{cos\left(\frac{\pi}{N}\right)}{sin\left(\frac{\pi(N-2)-A}{2N}\right)}\right).$$

Then the polygonal isoperimetric inequality reduces to

$$L(P) \geq \frac{3A}{\pi(n-6)} L(n, \frac{(n-6)\pi}{3}) = F(n, A).$$

So the polygonal isoperimetric inequality would say that $L(n, A) - F(n, A) \geq 0$, inside the range for A where $\frac{2\pi}{N^2\sqrt{3}} \leq A \leq \frac{N-6}{3}\pi$. Now to construct the counterexample, consider the case where $N = 70$ and $A = 64$ (which is within the bounds supplied above). Then

$$\begin{aligned} L(70, 64) &\approx 73.1426 \\ F(70, 64) &\approx 73.167 \\ L(70, 64) - F(70, 64) &\approx -0.0244389 \end{aligned}$$

So the polygonal isoperimetric inequality does not always hold. Larger differences between $F(n, A)$and $L(n, A)$ can be found for larger values of N and A. □

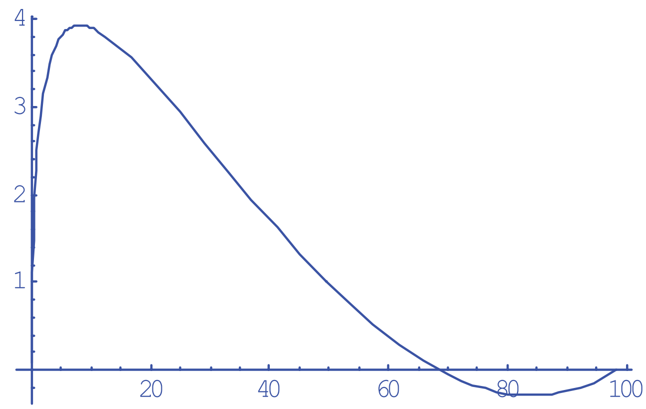

Figure 7. A mathematica generated plot showing the Polygonal Isoperimetric Inequality failing for large n.

6. The Honeycomb Conjecture on Compact Hyperbolic Manifolds

We assume that Conjecture 5.1 holds and prove the honeycomb conjecture on compact hyperbolic manifolds. We also assume that the values of b and c for which it holds can be bounded above by 2.1 and 0.08, respectively, for all $7 \leq N \leq 12$. While these upper bounds may not be precise, the proof should still hold with slightly different upper bounds.

Proposition 6.1. *Assume Conjecture 5.1. If M is a compact hyperbolic manifold such that the shortest non-contractible curve on the surface of M has length greater than $k_N = (P_N + A_N b - c(2 - N))$, and M admits a tiling by five or more regular N-gons where $7 \leq N \leq 12$, then any partition of the manifold into regular N-gons meeting in threes is perimeter minimizing.*

Proof. Consider a perimeter-minimizing partition on M. Let R be a component of this partition. We will first show that the Polygonal Isoperimetric Inequality (5.1) applies to all possible R. If R has a boundary component which is not contractible, then this boundary component has length $L_1 > k_n$. So,

$$\begin{aligned} L(R) > L_1 &> (P_N + bA_N - c(2 - N)) \\ &> (A(R)/A_N)P_N - bX(R) - c(N(R) - N), \end{aligned} \tag{18}$$

which means the Polygonal Isoperimetric Inequality holds for R.

Otherwise, all of the boundary components of R are contractible. Assume R has more than one boundary component. Then we can slide one of the boundary components until it intersects another boundary component or itself at some point. This partition would also be minimal but it would have a vertex at the point of intersection with valence greater than or equal to four, contradicting regularity. Hence the boundary of R is one contractible curve.

If R is contractible, then it is a disk, and we can apply the Polygonal Isoperimetric Inequality. If R is not contractible, then the complement of R must be since the boundary of R is contractible. Since the partition of M by regular N-gons requires at least five regions, we know that $A(R^C) \geq 4A_N$. Since R^C is a disk, by the isoperimetric inequality (Prop 2.2) $L(R^C) > \sqrt{4A_N(4\pi + 4A_N)}$. We claim that $\sqrt{4A_N(4\pi + 4A_N)} > P_N + bA_N - c(2-N)$. To show this we overestimate b and c. We want to show

$$\sqrt{4A_N(4\pi + 4A_N)} > P_N + 2.1A_N - 0.08(2-N). \tag{19}$$

Substituting in for P_N and A_N gives,

$$\sqrt{4\frac{\pi(N-6)}{3}\left(4\frac{\pi(N-6)}{3}\right)} > 2Ncosh^{-1}\left(\frac{\cos(\frac{\pi}{n})}{\sin(\frac{\pi}{3})}\right) + 2.1\frac{\pi(N-6)}{3} - 0.08(2-N).$$

Explicit evaluation at the values $7 \leq N \leq 66$ shows that the above inequality holds in these cases.

So, because $L(R) = L(R^C)$,

$$\begin{aligned} L(R) = L(R^C) &\geq P_N + bA_N - c(2-N) \\ &\geq \frac{A(R)}{A_N}P_N - bX(R) - c(N(R) - N), \end{aligned} \tag{20}$$

which implies the Polygonal Isoperimetric Inequality holds for R.

We can then apply the Polygonal Isoperimetric Inequality to all components in the partition. The proof now proceeds similar to Hales's proof [3] of the Euclidean case. Averaged over the manifold, $A(P)/A_N \leq 1$, the bulges must cancel out, so $bX(P) = 0$ and by Euler's Formula, $n - N \leq 0$ ([7], Lemma 2.1), so

$$L(P) \geq \frac{A(P)}{A_N}P_N - bX(P) - c(n-N) \geq P_N$$

Since a tiling or regular N-gons has exactly this average perimeter, it is a minimizer. □

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