

Isoperimetric Problems on the Line with Density $|x|^p$

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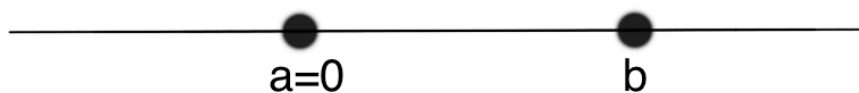
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Abstract. On the line with density $|x|^p$, we prove that the best single bubble is an interval with endpoint at the origin and that the best double bubble is two adjacent intervals that meet at the origin.

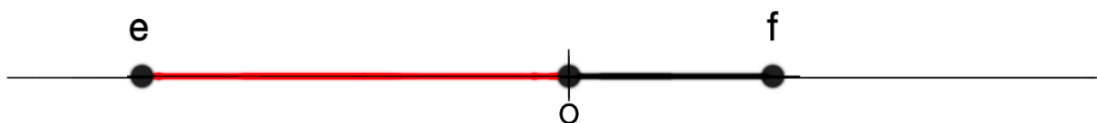
1. Introduction.

In the past dozen years there has been a surge of interest in manifolds with density, partly because of their role in Perelman's 2006 proof of the Poincaré Conjecture. We consider isoperimetric problems on the line with density $|x|^p$ and prove the following single and double bubble theorems:

Single Bubble Theorem (2.3). *On the line with density $|x|^p$ ($p > 0$), the least-perimeter region with given mass is an interval with one endpoint at the origin.*



Double Bubble Theorem (3.4). *On the line with density $|x|^p$, the least-perimeter way to enclose and separate two given masses is two adjacent intervals that meet at the origin.*



Density. Density is used to weight both perimeter and length. With density $|x|^p$, each boundary point b contributes $|b|^p$ to the perimeter, and the mass of an interval (a, b) is

$$\int_a^b |x|^p dx.$$

For example, the region of Figure 1.1 has perimeter $a^p + b^p + c^p$ and mass

$$\int_b^a x^p + \int_0^c x^p = \frac{a^{p+1} - b^{p+1} + c^{p+1}}{p+1}.$$

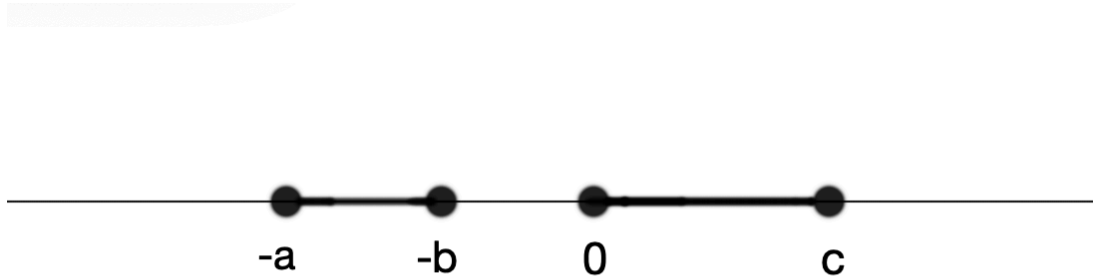


Figure 1.1. A region with (weighted) perimeter $a^p + b^p + c^p$ and mass

$$\int_b^a x^p + \int_0^c x^p.$$

The Black and Red regions of Figure 2.1 have perimeter $a^p + b^p + c^p + d^p + e^p$ and Black mass $\int_a^b |x|^p dx$ and Red mass $\int_b^c |x|^p dx + \int_d^e |x|^p dx$. Note that the common boundary point counts once.

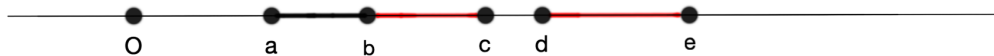


Figure 2.1. A Black and a Red region with total perimeter $a^p + b^p + c^p + d^p + e^p$, and Black mass $\int_a^b |x|^p dx$ and Red mass $\int_b^c |x|^p dx + \int_d^e |x|^p dx$.

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A finite-perimeter region on the line with unit density must consist of finitely many intervals. With density $|x|$, there is the possibility of countably many intervals converging to the origin.

History. The single bubble with our density $|x|^p$ was proved to be a sphere through the origin in \mathbf{R}^n for $n = 2$ by Dahlberg et al. [Da] and for $n > 2$ by Boyer et al. [Bo]. For this density the double bubble is new, although it is known in Euclidean and other spaces (see [Mo]). It was earlier studied at the 2017 Texas State Honors Summer Math Camp, but that work was not completed. The double bubble for density $|x|^p$ remains completely opened for higher dimensions.

Proofs. Our proofs employ elementary comparisons, without even any recourse to stability theory. For the single bubble, it follows from Rosales et al. [Ro, Thm. 4.14(iii)] that the minimizer is an interval containing the origin, but our proof does cite that fact.

A finite-perimeter region on the line with unit density must consist of finitely many intervals. With density $|x|$, there is the possibility of countably many intervals converging to the origin.

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2. The Best Single Region

Our main Theorem 2.3 identifies the least-perimeter single region of given mass on the line with density $|x|^p$ ($p > 0$). We start with a geometric proof that works only for $p = 1$.

2.1. Proposition. *On the line with density $|x|$, the least-perimeter single interval of given mass is an interval with one endpoint at the origin.*

Proof. We consider two cases according to whether the interval (a, b) contains the origin. If (a, b) does not contain the origin, we may assume that $0 \leq a < b$, as in Figure 2.1.

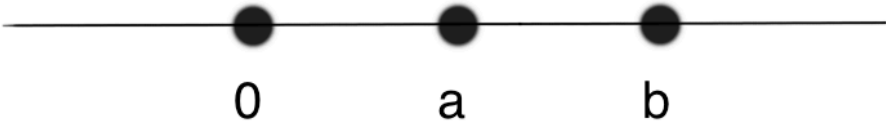


Figure 2.1. In the first case, the origin lies outside the interval, which has perimeter $a + b$.

The perimeter equals $a + b$. An interval $(0, b')$ of the same mass beginning at the origin, as in Figure 2.2, has perimeter $b' \leq b \leq a + b$, with equality only if we had that interval to start with.

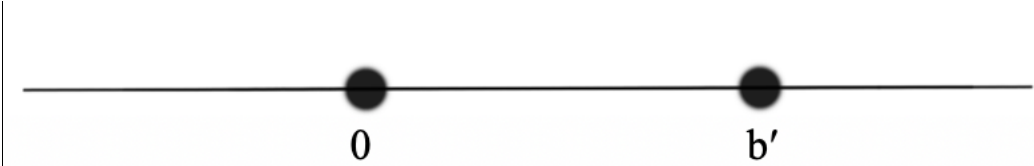


Figure 2.2. An interval of the same mass starting at the origin has perimeter $b' \leq b \leq a + b$.

On the other hand, as in Figure 2.3, consider an interval $(-a, b)$ containing the origin and compare it to the interval $(0, b')$ of the same mass, obtained by replacing $(-a, b)$ with an interval (b, b') of the same mass. Since the latter has larger average density, $b' - b < a$, so that $b' < a + b$. Therefore $(0, b')$ has less perimeter than $(-a, b)$.

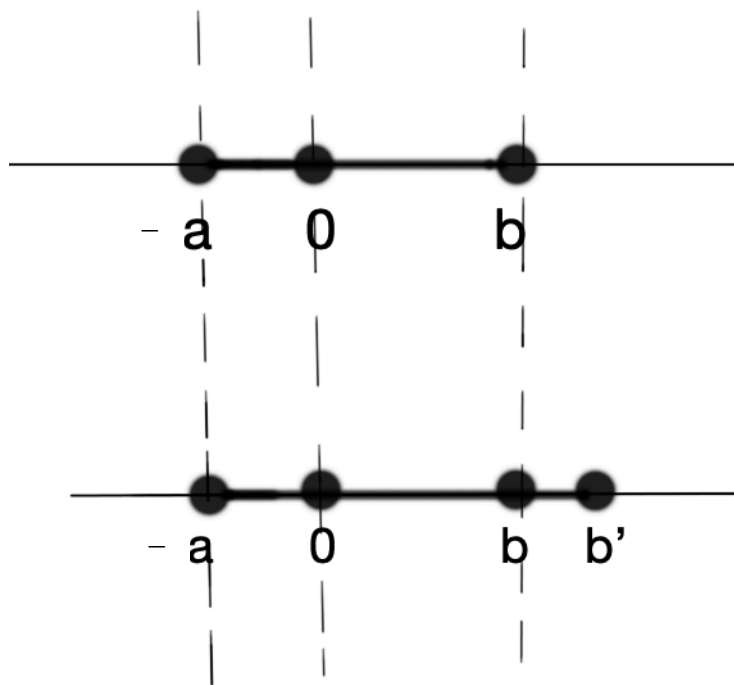


Figure 2.3. In the second case, the interval $(-a, b)$ contains the origin.

2.2. Remark. We provide another argument for the second case of the preceding proof (Figure 2.2). The mass of $(-a, 0)$ and $(0, b)$ may be computed as the length times the average density, yielding total mass

$$m = a(a/2) + b(b/2) = \frac{a^2 + b^2}{2},$$

so that

$$a^2 + b^2 = (\sqrt{2m})^2.$$

On this quarter circle, as in Figure 2.4, the perimeter $a + b$, represented by two sides of the pictured triangle, always exceeds the third side $\sqrt{2m}$, with equality only if a or b is 0, i.e., when the interval has endpoint at the origin.

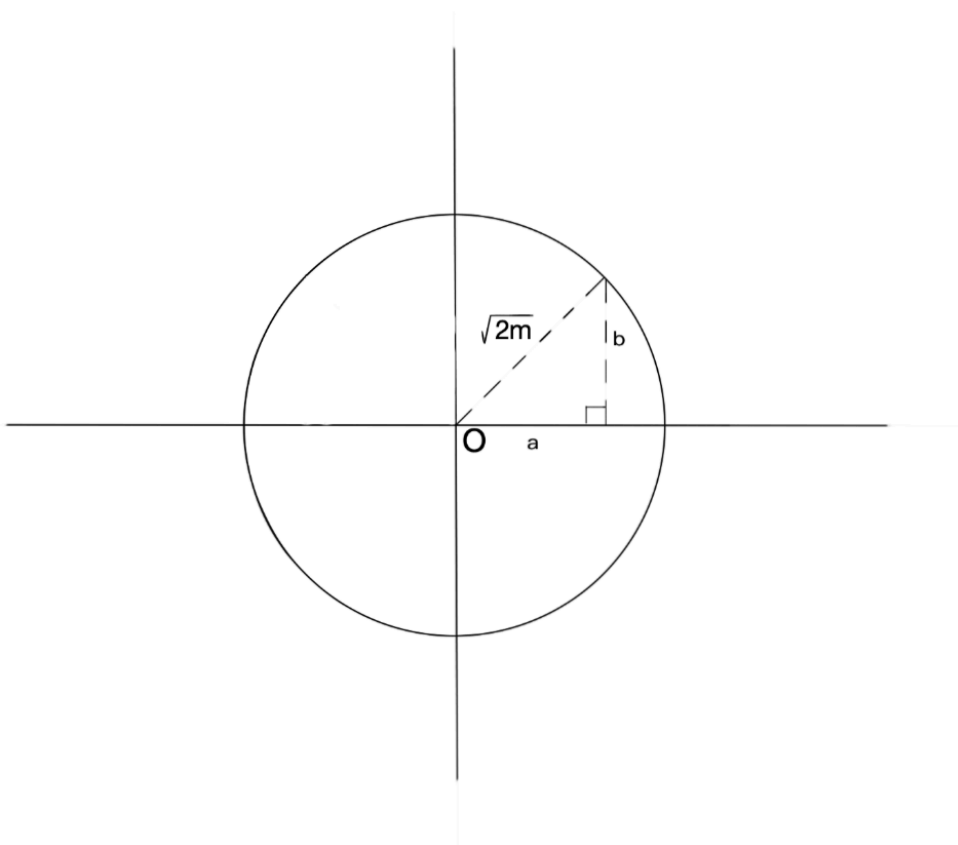


Figure 2.4. Given $a^2 + b^2 = (\sqrt{2m})^2$, $a + b$ is minimized when a or b is 0.

We now give our main single bubble theorem. Our first, geometric proof works just for density $|x|$. Our second, algebraic proof works for density $|x|^p$.

2.3. Theorem. *On the line with density $|x|^p$ ($p > 0$), the least-perimeter region with given mass is an interval with one endpoint at the origin.*

Proof for density $|x|$. Consider a region with a given mass and finite perimeter, perhaps consisting of many intervals as shown in Figure 2.5. Replace the mass to the right of the origin with a single interval $(0, b)$. The new perimeter b no greater, indeed is less than or equal to the cost of the original right-most endpoint.

Similarly replace the mass to the left of the origin by a single interval $(-a, 0)$. We have thus produced a single interval $(-a, b)$ of no greater perimeter, with equality only if the original region was that interval. Finally, by Proposition 2.1, a single interval $(-a, 0)$ or $(0, b)$ is best.

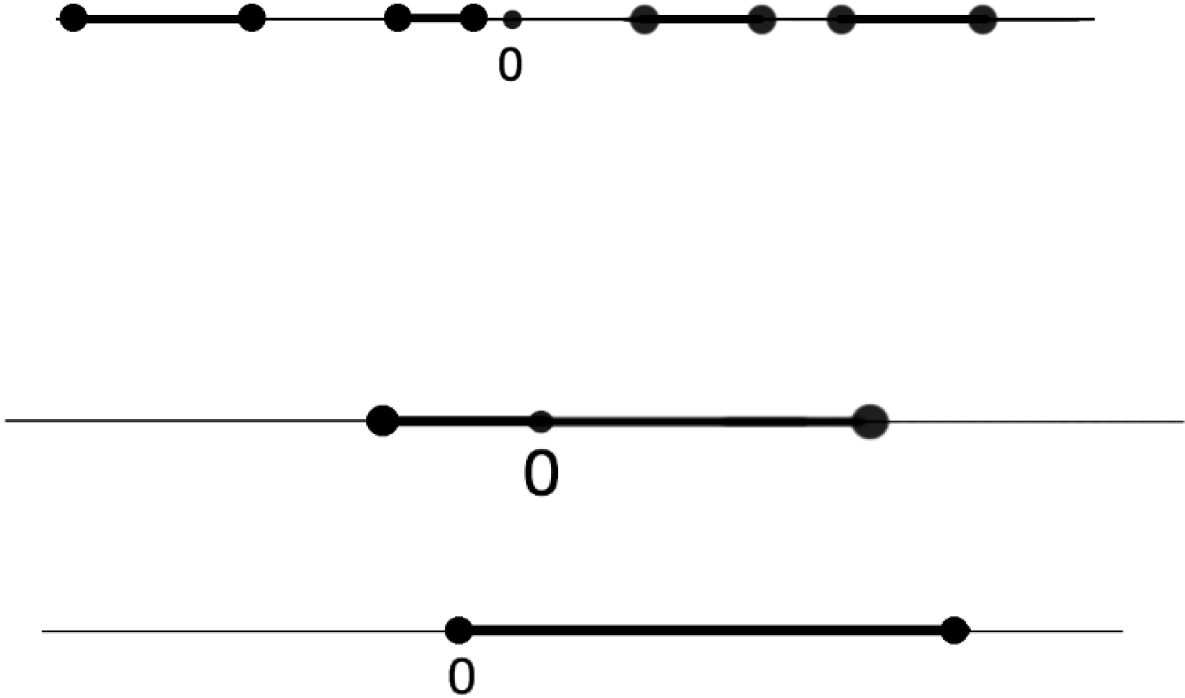


Figure 2.5. Replacing the portions of a given region to the left and right of the origin with single intervals reduces perimeter. Then by Proposition 2.1 a single interval with endpoint at the origin is best.

Proof for density $|x|^p$. Consider any region of finite perimeter. It consists of countably many intervals, the origin the only possible limit point. If an interval contains the origin, split it into two intervals. Denote the intervals to the right of the origin by (a_i, b_i) and the intervals to the left of the origin by $(-c_i, -d_i)$. Note that the mass of (a, b) is the integral from a to b of x^p , namely $(b^{p+1}-a^{p+1})/(p+1)$. Hence the total mass m satisfies

$$(p+1)m = \sum(b_i^{p+1}-a_i^{p+1}) + \sum(c_i^{p+1}-d_i^{p+1}),$$

and the total cost is $\sum a_i^p + b_i^p + c_i^p + d_i^p$.

Compare with the single interval $(0, b)$ with the same mass $m = b^{p+1}/(p+1)$ and perimeter b^p . Since they have the same mass,

$$\sum(b_i^{p+1}-a_i^{p+1}) + \sum(c_{ip+1}-d_{ip+1}) = b^{p+1}.$$

Therefore,

$$\sum(b_i^{p+1}+a_i^{p+1}) + \sum(c_i^{p+1}+d_i^{p+1}) \geq b^{p+1}.$$

Now by Lemma 2.5, the total perimeter satisfies

$$\sum(b_i^p+a_i^p) + \sum(c_i^p+d_i^p) \geq b^p,$$

the perimeter of $(0, b)$, with equality only if we started with a single interval with endpoint at the origin.

2.5. Lemma. For $p > 0$, suppose $\sum a_i^{p+1} \geq b^{p+1}$.

Then
$$\sum a_i^p \geq b^p,$$

with equality only if there is just one nonzero a_i .

Proof. Let
$$A_i = a_i^p$$

$$u = (p+1)/p > 1.$$

Now the lemma reduces to the standard fact that

$$(\sum A_i)^u \geq \sum A_i^u.$$

Indeed, after normalizing to $\sum A_i^u = 1$, the result is trivial.

3. The Best Double Region

Our main Theorem 3.4 will provide the least-perimeter way to enclose and separate *two* given masses (“Black” and “Red”) on the line with density $|x|^p$ ($p > 0$). Our first proof works just for density $|x|$ start with a proposition about the half-line that will reduce candidates on the whole line to those consisting of two, three, or four adjacent intervals, in turn handled by Propositions 3.2 and 3.3.

3.1. Proposition. *On the positive axis $\{x > 0\}$ with density $|x|$, given two regions, there are two adjacent intervals with one endpoint at 0 with the same masses and no more perimeter.*

Proof. Consider two regions such as the Black and Red intervals of Figure 3.1. We may assume that the last interval is Red. In comparison, take two adjacent Black and Red intervals at the origin with the same masses as the original regions, as in Figure 3.1. The first boundary point contributes no more perimeter than the last Black boundary point of the original, and the second boundary point contributes no more perimeter than the last Red boundary point of the original. Therefore, those adjacent intervals of the same masses have no more perimeter as claimed.

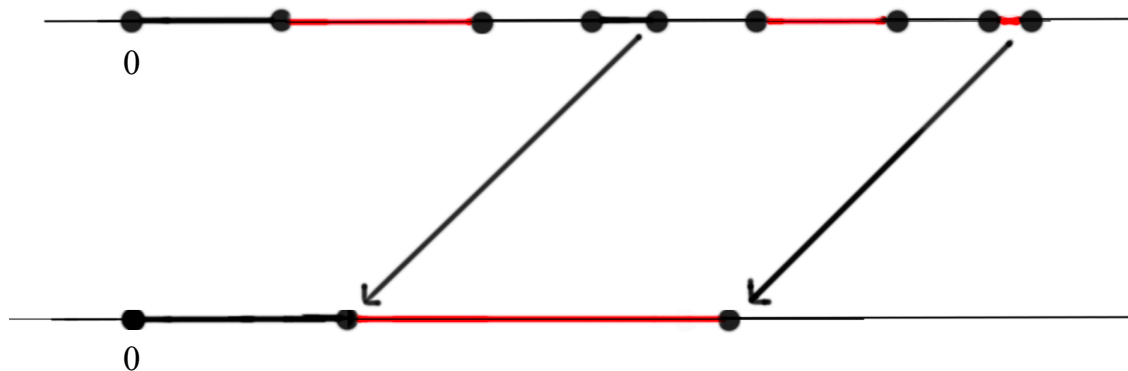


Figure 3.1. Two intervals are better than many.

3.2. Proposition. *On the line with density $|x|$, when enclosing and separating “Black” and “Red” masses, two intervals Red and Black meeting at the origin require less perimeter than any other adjacent Red-Black-Red or Red-Black with the origin inside Black.*

Proof. First, we reduce the Red-Black-Red case to the Red-Black case. As shown in Figure 1, the perimeter for the Black region is $b + c$. By symmetry, we may assume that $c < b$.

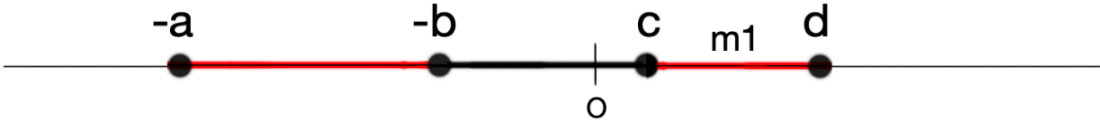


Figure 3.2. Adjacent Red-Black-Red with the origin inside Black.

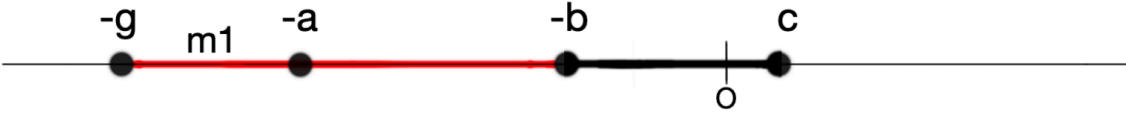


Figure 3.3. Move the mass of (c, d) to the left of $(-a, -b)$.

As in Figure 3, move the mass of the interval (c, d) to the left to form an interval $(-g, -a)$. Since we have assumed that $c < b$, the interval $(-g, -a)$ is farther from the origin than (c, d) , and hence its length is less: $g - a < d - c$. Therefore, the new Red-Black perimeter, $b + c + g$ is less than the original Red-Black-Red perimeter $a + b + c + d$.

Finally, we compare Red-Black with the origin inside Black with the origin at their meeting point, as in Figure 3.4. By Proposition 2.1, $f \leq b + c$ and $e < g$. Therefore, the two intervals that meet at the origin are better as claimed.

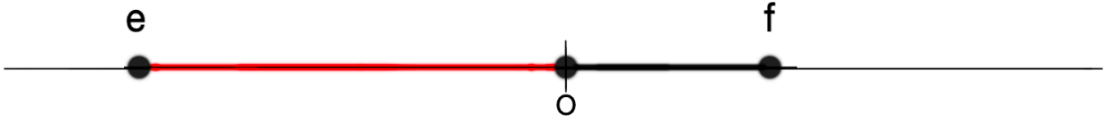


Figure 3.4. Meeting at the origin is best.

3.3. Proposition: *On the line with density $|x|$, when enclosing and separating “Black” and “Red” masses, two intervals meeting at the origin have less perimeter than adjacent Red-Black-Origin-Red-Black.*

Proof. By symmetry, we may assume that $c < b$. As in Figure 3.5, we show that exchanging the outside masses reduces perimeter and yields the desired two intervals meeting at the origin.

Since $c < b$, the interval $(-a', -b)$ is farther from the origin than (c, d) , and hence the length is less: $a' - b < d - c$. Since $c < b$, the interval $(0, d')$ has less perimeter than $(-a, 0)$: $d' < a$.

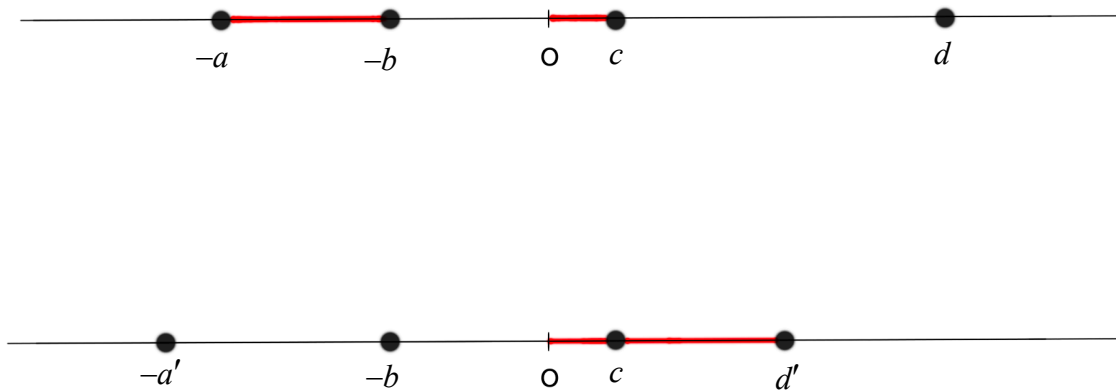


Figure 3.5. Exchanging the outside masses reduces perimeter.

As a result, $a' + d' < a + b - c + d$. Therefore, the new perimeter for both regions $a' + d'$ is less than the original perimeter $a + b + c + d$.

We now give our main double bubble theorem. Our first, geometric proof works just for density $|x|$. Our second, algebraic proof works for density $|x|^p$.

3.4. Theorem. *On the line with density $|x|$, the least-perimeter way to enclose and separate two given masses is two adjacent intervals that meet at the origin.*

Proof for density $|x|$. Consider two regions of finite perimeter and given masses. To each half of the line meeting both regions, apply Proposition 3.1 to reduce to the case of two adjacent intervals with one endpoint at the origin. To each half of the line meeting just one region, a single interval with one endpoint at the origin is best. We may assume the resulting candidate consists of adjacent intervals of the “Black” and “Red” regions of one of the following three types:

- a. Red-Black with the origin inside Black,
- b. Red-Black-Red with the origin inside Black,
- c. Red-Black-Origin-Red-Black.

By Proposition 3.2 and 3.3, two adjacent intervals that meet at the origin is best.

Proof for density $|x|^p$. Consider two regions (“Black” and “Red”) of finite perimeter. Each consists of countably many intervals, the origin the only possible limit point. If an interval contains the origin, split it into two intervals. Denote the Black intervals by (a_i, b_i) and $(-c_i, -d_i)$, the Red intervals by (e_i, f_i) and $(-g_i, -h_i)$. Note that the mass of (a, b) is the integral from a to b of x^p , namely $(b^{p+1} - a^{p+1}) / (p+1)$. Hence the total Black mass m_B satisfies

$$(p+1)m_B = \sum(b_i^{p+1} - a_i^{p+1}) + \sum(c_i^{p+1} - d_i^{p+1}),$$

and the Red mass m_R satisfies

$$(p+1)m_R = \sum(f_i^{p+1} - e_i^{p+1}) + \sum(h_i^{p+1} - g_i^{p+1}).$$

The total perimeter is at least $\sum b_i^p + c_i^p + d_i^p + f_i^p + g_i^p$, with equality only for two intervals that meet at the origin. Note that by using just the outside endpoint of each interval, we avoid double counting endpoints common to adjacent Black and Red intervals.

Compare with two intervals $(-r, 0)$ and $(0, s)$ with the same mass $m_B = r^{p+1} / (p+1)$ and mass $m_R = s^{p+1} / (p+1)$ and cost $b^p + a^p$. Since they have the same mass,

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$$\sum(b_i^{p+1}-a_i^{p+1}) + \sum(c_i^{p+1}-d_i^{p+1}) = r^{p+1}$$

$$\sum(f_i^{p+1}-e_i^{p+1}) + \sum(g_i^{p+1}-h_i^{p+1}) = s^{p+1}$$

Therefore,

$$\sum b_i^{p+1} + \sum c_i^{p+1} \geq r^{p+1}$$

$$\sum f_i^{p+1} + \sum g_i^{p+1} \geq s^{p+1}$$

Now by Lemma A,

$$\sum b_i^p + \sum c_i^p \geq r^p$$

$$\sum f_i^p + \sum h_i^p \geq s^p$$

Hence the total cost is at least

$$\sum b_i^p + \sum c_i^p + \sum f_i^p + \sum g_i^p \geq r^p + s^p$$

$(-r, 0)$ and $(0, s)$, with equality only if we started with two intervals meeting at the origin.

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Biographical Sketch

This paper is the work of an Institute of Advanced Research winter camp course in Shanghai for high school students, under the direction of Frank Morgan.

Juiyu Huang from Nanjing International School likes mathematics, natural sciences, and badminton.

Xinkai Qian from Vermont International Academy, CISK, loves science. In his school's "RC pilots" club, he builds model planes and flies them. He really enjoyed doing mathematical research in a group and hopes to do more in the future.

Yiheng Pan, a senior at Hangzhou Foreign Language School, has the spirit of exploration and a passion for geometry.

Mulei Xu, a grade 12 student from Nanjing Experimental School, loves geometry, algebra, and astrophysics.

Lu Yang, a junior at Sun Yat-sen University, served as a teaching assistant. She wants to use statistics to predict the future.

Junfei Zhou from Chongqing Tsinghua High School loves mathematics, likes physics, and has a certain interest in chemistry and biology.