On Construction Using Construction Tools of Finite Size

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I wrote this article in Thai as a high school student in 2011. The present article is an English translation with minor revisions.

0. Introduction and Definitions

In a geometric construction, we often assume that our construction tools (usually a compass and a straightedge) have infinite size. But in practice, our construction tools have finite size, leading to obstacles in the construction. For example, given two points about 10 meters apart, if we would like to use a compass and a straightedge to find the midpoint, there will be a problem that we cannot draw a line segment connecting the two points. This leads to the question of whether such a construction is possible. However, this document will show that we can indeed carry out the construction, as we will later show. But first, the meaning of the word "size" of the tools that we mentioned is not very clear, so we need to define some terminology.

First, we view a *construction* as a mapping of a given geometric shape into the geometric shape that we want to construct (a *geometric shape* means a set of points on the plane). So, a construction tool must have a *construction function*, which is the mentioned mapping, and it must have the *domain* of the tool, which is the domain of that function. Some tools may be able to do many kinds of constructions. For example, a square piece can be used as a straightedge, to create a right angle, etc. So it is necessary to view this kind of tool as a *set of tools*, that is, being comprised of many tools. A "compass and straightedge" is also clearly a set of tools.

The construction tools that we are interested in here must also be *shrinkable*[1] tools. This means that if a geometric shape P is in the domain of the tool, then any geometric shape Q that is obtained by shrinking P where the origin is the center of shrinking must also be in the domain of the tool. And if we let f be the construction function of that tool, then $f(Q)$ has to be the geometric shape obtained by shrinking $f(P)$ with the same ratio as the shrinking of P into Q where the origin is the center of shrinking.

We call a shrinkable construction tool a *construction tool of infinite size* if and only if whenever a geometric shape P is in the domain of the tool, any geometric shape obtained by enlarging P must also be in the domain of the tool. A shrinkable construction tool that is not of infinite size is called a *construction tool of finite size*. A set of shrinkable construction tools has *infinite size* if all construction tools in the set have infinite size, and has *finite size* if it does not have infinite size.

Two shrinkable construction tools T_1, T_2 that have construction functions f_1, f_2, respectively, are called *of the same kind* if and only if for any geometric shape P in the domain of T_1, there is a geometric shape Q in the domain of T_2, and for any geometric shape Q in the domain of T_2, there is a geometric shape P in the domain of T_1, with the property that P and $f_1(P)$ can be shrank or enlarged into Q and $f_2(Q)$.

[1] Translator's Note: The original term is "continuous." Since this may be confusing, I take liberty in using a new term.

Two sets of construction tools are *of the same kind* if and only if we can pair up the tools in the two sets in a one-to-one and onto manner, so that the construction tools in each pair are of the same kind. For shrinkable construction tools T_1, T_2 that are of the same kind, we say that T_1 is *larger* than T_2 if and only if the domain of T_2 is a strict subset of the domain of T_1, and we say that T_1 is *smaller* than T_2 if and only if T_2 is larger than T_1. Observe that the fact that T_1 is neither larger nor smaller than T_2 does not lead to T_1, T_2 being the same construction tool.

Observe that, for shrinkable construction tools of the same kind, the larger one can always construct things that the smaller one can. The following theorems give some conditions under which a small construction tool is equivalent to a large construction tool in carrying out some kinds of constructions.

1. Preliminaries

First, it is not completely clear that one step of the construction in the theorem below (as well as in some other constructions) is doable. So we will propose a new axiom to support such a construction, as follows.

Definition. A *cluster* is a geometric shape that contains a non-degenerate closed disk as a subset.

Axiom 1. Let P be a cluster. We can always construct a point A so that it is inside P.

The following axiom allows us to join segments.[2]

Axiom 2. Given segments AB and BC where A, B, C are collinear, we can construct the segment AC.

2. Theorems

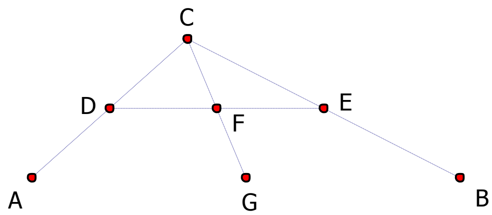
Theorem 1 (Constructing points). *Suppose that a set of shrinkable construction tools T can do all of the following things:*
(i) *Construct the midpoint of two given points whose distance is less than a certain constant;*
(ii) *Reflect a given point across a given point, where the distance between the two points is less than a certain constant.*
Then any construction of a finite number of points from a finite number of given points and the origin[3] *that can be done by a set of construction tools T' of the same kind as T can be done by T.*

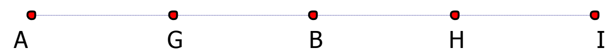
Proof. First, we say that a set of shrinkable construction tools has property $P(a)$ if and only if it can do (i), where the mentioned constant is a. And it has property $Q(a)$ if and only if it can do (ii), where the mentioned constant is a.

Lemma. *If a set of shrinkable construction tools T has properties* $P(a)$ *and* $Q(a)$ *for some positive real number a, then it also has properties* $P(2a)$ *and* $Q(2a)$.

[2] TN: Axiom 2 is added to fix a minor gap in Theorem 2.
[3] TN: "And the origin" is added to fix a minor gap. This is also added in the statement of Theorem 2.

Proof.

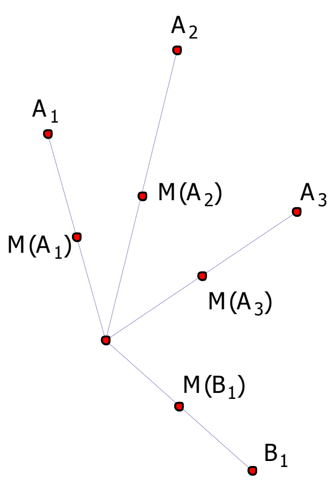




First we will show that T has property $P(2a)$. Let points A, B be less than $2a$ apart. We want to construct the midpoint of AB. By Axiom 1, we can construct a point C such that $AC < a$, $BC < a$. From property $P(a)$, we can construct the midpoint D of AC and the midpoint E of BC. Consider that $DE = ½\ AB < a$, so from property $P(a)$, we can construct the midpoint F of DE. Consider that, from the Triangle Inequality, $CF < CE + EF = ½\ BC + ½\ DE < a$. So from property $Q(a)$, we can reflect C across F to become G. We get that points D, E, F are enlarged to become A, B, G, respectively, with ratio 2 where C is the center of enlarging. Therefore, G is the midpoint of AB, as desired.

It remains to show that T has property $Q(2a)$. That is, we want to reflect A across B. From property $Q(a)$, we can reflect G across B to become H and reflect B across H to become I. We get that I is the point obtained by reflecting A across B, as desired. Therefore the lemma is true. □

By the lemma, we get that T has properties $P(a)$ and $Q(a)$ for all positive real numbers a. Consider the following. Let the finite number of given points be $A_1, A_2, \dots, A_n$ and the finite number of points we want to construct be $B_1, B_2, \dots, B_m$. Define the function M on the set of points on the plane as follows. For any point A, let $M(A)$ be the midpoint of the origin and point A. By property $P(a)$, we can construct $M(A_1), \dots, M(A_n)$, and so can construct $M^k(A_1), \dots, M^k(A_n)$ for all positive integers k.



By the definition of T and T' being of the same kind, there is a large enough positive integer k such that from $M^k(A_1), \dots, M^k(A_n)$, we can use T to construct $M^k(B_1), \dots, M^k(B_m)$. This is because in each step of the construction using T', there must be a shrinking or enlarging of that step that T can construct. And then we choose k large enough so that the step that results [from shrinking by a factor of 2^k][4] is small enough so that T can construct it, because T can construct small geometric shapes due to T being shrinkable.

Then, by the property $Q(a)$, we can enlarge $M^k(B_1), \dots, M^k(B_m)$ back into $B_1, B_2, \dots, B_m$, as desired. Therefore the theorem is true. □

Theorem 2 (Constructing points and segments). *Suppose that a set of shrinkable construction tools T can do all of the following things:*
(i) *Construct the midpoint of two given points whose distance is less than a certain constant;*

[4] TN: This phrase is added to clarify the argument.

(ii) *Reflect a given point across a given point, where the distance between the two points is less than a certain constant;*
(iii) *Construct the segment connecting two given points whose distance is less than a certain constant.*
Then any construction of a finite number of points and segments with endpoints[5] *from a finite number of given points*[6] *and the origin that can be done by a set of construction tools T' of the same kind as T can be done by T.*

Proof. First, we say that a set of shrinkable construction tools has property $P(a)$ if and only if it can do (i), where the mentioned constant is a. And it has property $Q(a)$ if and only if it can do (ii), where the mentioned constant is a. And it has property $R(a)$ if and only if it can do (iii), where the mentioned constant is a.

Lemma. *If a set of shrinkable construction tools T has properties* $P(a)$, $Q(a)$, $R(a)$ *for some positive real number a, then it also has properties* $P(2a)$, $Q(2a)$, $R(2a)$.

Proof. From the lemma of Theorem 1, we get that T has properties $P(2a)$ and $Q(2a)$. It only remains to show that it has property $R(2a)$.

Let points A, B be less than $2a$ apart. We want to construct the segment AB. From property $P(2a)$, we can construct the midpoint M of AB. And consider that $AM < a$, $BM < a$. So by property $R(a)$,[7] we can construct AM, BM, and so by Axiom 2, construct AB, as desired. Therefore the lemma is true. □

By the lemma, we get that T has properties $P(a)$, $Q(a)$, $R(a)$ for all positive real numbers a. [By Theorem 1, we can use T to construct the required points and endpoints of segments. Then the segments themselves can be constructed by the lemma.][8] Therefore the theorem is true. □

3. An Application

Corollary (Construction with a compass of finite radius and a straightedge of finite length). *A construction of a finite number of points and segments from a finite number of given points, lines, segments with endpoints, rays with endpoints, and circles that can be done with a compass of infinite radius and a straightedge of infinite length can be done with a compass of finite radius and a straightedge of finite length.*

Proof.[9] Let T be a set of a compass of infinite radius and straightedges of every finite length. First notice that the required construction can be done by T: every straight line that is produced as an intermediate step can be replaced by a long enough segment using T. Now observe that T is of the same kind as a set of a compass of finite radius and a straightedge of finite length. So we can apply Theorem 2.

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[5] TN: "With endpoints" is added to fix a minor gap. This is also added in the following corollary.
[6] TN: "Points, lines, segments, and rays" is changed to "points" to fix a minor gap.
[7] TN: The original text says $R(2a)$, which is a typographical error.
[8] TN: This part of the argument is rewritten due to changes in the theorem statement.
[9] TN: The entire proof is rewritten to accord with Theorem 2 and to fix some minor gaps.

Use two points to represent each line, segment, and ray, three points to represent each circle, and an arbitrary point as the origin. To construct these representative points, draw small segments to intersect each line, ray, and circle. Then from the given points and the representative points, we can use *T* to construct the desired points and segments. The endpoints of segments are automatically constructed since they must be constructed before the segments are constructed. By Theorem 2, a compass of finite radius and a straightedge of finite length can also carry out the construction. Therefore the corollary is true. □

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