Urinal Problem

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I solved this problem as a high school student in 2013, ignorant that Philippe Flajolet had already solved it at least asymptotically (http://algo.inria.fr/libraries/autocomb/fatmen-html/fatmen1.html) in 1997. Flajolet stated that the original problem dated back to at least 1962.

Problem. There are n urinals arranged in a straight line. One by one, people choose their own spots in front of the urinals, where each person randomly chooses a urinal that is not adjacent to any previously occupied urinal, until there are no urinals that are not adjacent to any occupied urinal left. Find the expected value of the number of people in front of the urinals.

Answer. The expected value of the number of people is

$$f(n) = \frac{n+1}{2} - \frac{n+3}{4}\left(A(n+2) + A(n+3)\right),$$

where $A(n) = \sum_{k=0}^{n} \frac{(-2)^k}{k!}$ for all natural numbers n.

Because $\lim_{n\to\infty} A(n) = e^{-2}$, we can approximate

$$f(n) \approx \left(\frac{1-e^{-2}}{2}\right)n + \frac{1-3e^{-2}}{2},$$

or approximating further up to two decimal points,

$$f(n) \approx 0.43n + 0.30.$$

We find that when $n \geq 4$, the approximations give satisfactory results as shown in the following table.

n	$f(n)$	$\left(\frac{1-e^{-2}}{2}\right)n + \frac{1-3e^{-2}}{2}$	$0.43n + 0.30$
4	2.0000	2.0263	2.02
5	2.4667	2.4587	2.45
6	2.8889	2.8910	2.88
7	3.3238	3.3233	3.31
8	3.7556	3.7557	3.74
9	4.1880	4.1880	4.17
10	4.6203	4.6203	4.60
20	8.9436	8.9436	8.90
30	13.2670	13.2670	13.20
40	17.5903	17.5903	17.50
50	21.9136	21.9136	21.80

Solving the problem

Let $f(n)$ be the expected value of the number of people where there are n urinals. Construct a recurrence relation for $f(n)$ by dividing into cases where the first person occupies each of the n spots. We get

$$f(n)=1+\frac{f(n-2)+f(n-1)+\left(f(1)+f(n-2)\right)+\left(f(2)+f(n-3)\right)...+f(n-2)}{n},$$

$$f(n)=1+\frac{2}{n}\left(f(1)+f(2)+...+f(n-2)\right),$$

for all natural numbers $n\geq 3$. Next, substitute n with $n+1$ to get

$$f(n+1)=1+\frac{2}{n+1}\left(f(1)+f(2)+...+f(n-1)\right).$$

Using this equation combined with the previous equation yields the recurrence equation

$$(n+1)f(n+1)=nf(n)+2f(n-1)+1,$$

where $f(1)=1, f(2)=1$. Observe that $\frac{n}{3}$ is a solution to this recurrence equation. Let $f(n)=\frac{n}{3}+g(n)$. We get

$$(n+1)g(n+1)=ng(n)+2g(n-1),$$

where $g(1)=\frac{2}{3}, g(2)=\frac{1}{3}$. Observe that $n+3$ is a solution to this recurrence equation. Let $g(n)=(n+3)h(n)$. We get

$$(n+1)(n+4)h(n+1)=n(n+3)h(n)+2(n+2)h(n-1),$$

where $h(1)=\frac{1}{6}, h(2)=\frac{1}{15}$. Consider that the sums of the coeffcients on the two sides of the equation are equal, that is $(n+1)(n+4)=n(n+3)+2(n+2)$. Therefore the equation can be written in terms of differences as

$$h(n+1)-h(n)=-\frac{2(n+2)}{(n+1)(n+4)}\left(h(n)-h(n-1)\right).$$

We get that the value $h(n)-h(n-1)$ can be written as a product

$$h(n)-h(n-1)=(-1)^{n-1}\frac{2^n(n+1)}{(n+3)!},$$

where $h(0)=0$. Hence

$$h(n)=\sum_{k=1}^{n}(-1)^{k-1}\frac{2^k(k+1)}{(k+3)!}.$$

Define the function $A(n)=\sum_{k=0}^{n}\frac{(-2)^k}{k!}$ for all natural numbers n. We get

$$\begin{aligned}
h(n) &= \sum_{k=1}^{n}(-1)^{k-1}\frac{2^k\left((k+3)-2\right)}{(k+3)!} \\
&= \sum_{k=1}^{n}(-1)^{k-1}\frac{2^k}{(k+2)!}+\sum_{k=1}^{n}(-1)^{k}\frac{2^{k+1}}{(k+3)!} \\
&= -\frac{1}{4}\left(A(n+2)-1+2-2\right)-\frac{1}{4}\left(A(n+3)-1+2-2+\frac{4}{3}\right) \\
&= \frac{1}{6}-\frac{1}{4}A(n+2)-\frac{1}{4}A(n+3).
\end{aligned}$$

Substituting the value of $h(n)$ back into the equation $f(n)=\frac{n}{3}+(n+3)h(n)$, we obtain

$$f(n)=\frac{n+1}{2}-\frac{n+3}{4}\left(A(n+2)+A(n+3)\right),$$

as desired.

By Natso, 14 Mar 2013 (Pi Day)
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