Irrational Exponents in Fermat's Last Theorem. The $n > 2$ case.

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Abstract

In this paper we aim to find irrational exponents greater than 2 solving the Fermat's equation. We will find that a class of exponents can be derived from the solution of algebraic equations and so we will name them Algebraically Derived Ratio of Irrational (or ADRI) numbers. We will then further expand the class of ADRI numbers to cover all the solution exponents for any Fermat's equation.

It is well known that Fermat conjectured that no three positive integers x, y, z can satisfy the equation

$$x^n + y^n = z^n \tag{1}$$

for any integer $n > 2$. And this conjecture has been proved true by Wiles [7] after the useless efforts of uncountable matematicians.

Some studies [1-6] carried out on this equation demonstrated that the Fermat's theorem still holds for rational $n > 2$, so the equation

$$x^{\frac{k}{m}} + y^{\frac{k}{m}} = z^{\frac{k}{m}} \tag{2}$$

is never satisfied for $|k| > 2$.

We put an absolute value around k because, in the same papers, it is demonstrated that the solutions for a negative exponent are directly related to the solutions for its absolute value. So, if you don't have a solution for a positive n you'll have no solution also for $-n$.

Expanding the range of n over the set of the reals, Sun Ho [1] and Geeta S Kawale [2] found solutions when the equation is of this kind:

$$1^n + b^n = \left(b^2\right)^n. \tag{3}$$

These equations can be easily solved putting $s = b^n$ as they are solved by the algebraic equation:

$$1 + s = s^2 \tag{4}$$

Kawale finds $s = (\sqrt{5} + 1)/2$ and consequently $n = \log_b s$. For every integer b we can find the corrispective exponent that solves (4) but, as $s < 2$, all the exponents we find this way are less than 1.

So Fermat's equation confirms its evasive nature when trying to find exponents greater than 2 but we can easily demonstrate that it has, indeed, such solutions: Dominico Aiello [1] found a solution for the equations of the kind:

$$k^n + k^n = m^n. \tag{5}$$

To solve them you just have to notice that

$$2k^n = m^n \Leftrightarrow 2 = \left(\frac{m}{k}\right)^n \Leftrightarrow n = \log_{m/k} 2.$$

By choosing appropriate values for m and k (for example $m = 5$ and $k = 4$) we finally find:

$$4^n + 4^n = 5^n \quad n = \log_{5/4} 2 \approx 3.10628$$

the first equation with an exponent greater than 2, too bad we have $x = y$.

Kawale [2] was very near to find closed-form exponents for equation (3) which are greater than 2, we will demonstrate it in the following

Proposition 1 The equation

$$\left(k^2\right)^n + (km)^n = \left(m^2\right)^n \tag{6}$$

for k, m integers, has solutions for

$$n = \log_{m/k} s \tag{7}$$

where $s = (\sqrt{5} + 1)/2$

Proof To prove this result it is enough to divide both sides by k^2 obtaining an equation exactly equivalent to (3) where $b = k/m$. In the same way we solved (3), the solution just follows.

So to find $n > 2$ we just have to choose $\frac{\log s}{\log m/k} > 2$ or, equivalently,

$$\log m/k < \frac{\log s}{2} \Leftrightarrow \frac{m}{k} < \sqrt{\frac{\sqrt{5}+1}{2}} \approx 1.2720$$

Now the question is: "Can't we do any better? Can we find other classes for which it is possible to find an exponent solving Fermat's equation? Definitively, given three positive integers x, y, z, can we find the n that solves (1)?"

Let's start with the second question in this:

Proposition 2 The equation (1) has a closed form for the exponent when, chosen integers a, b coprime and integers m and k coprime with $m \neq 1$ such that $0 < m < k \leq 4$, we let $x = a^k$, $y = a^{k-m}b^m$, $z = b^k$. In this case:

$$n = \frac{\log(1+s)}{k \log b/a} \tag{8}$$

where s is a real solution of the algebraic equation:

$$(1+s)^m = s^k. \tag{9}$$

k	m	s
2	1	$\frac{\sqrt{5}+1}{2} \approx 1.6180$
3	1	$\frac{1}{3}\left(\sqrt[3]{\frac{27-3\sqrt{69}}{2}}+\sqrt[3]{\frac{27+3\sqrt{69}}{2}}\right) \approx 1.3247$
3	2	$\frac{1}{3}\left(1+\sqrt[3]{\frac{47-3\sqrt{93}}{2}}+\sqrt[3]{\frac{47+3\sqrt{93}}{2}}\right) \approx 2.1479$
4	1	too long to be reported ≈ 1.2207
4	3	too long to be reported ≈ 2.6297
5	4	$\frac{2}{3}+\frac{1}{3}\left(\sqrt[3]{\frac{97-3\sqrt{69}}{2}}+\sqrt[3]{\frac{97+3\sqrt{69}}{2}}\right) \approx 3.0288$

Table 1: Values for s

Proof Let's start with the Fermat's equation, as $z \neq 0$ we can divide by z^n:

$$x^n + y^n = z^n \Leftrightarrow \left(\frac{x}{z}\right)^n + \left(\frac{y}{z}\right)^n = 1 \Leftrightarrow \left[1+\left(\frac{y}{x}\right)^n\right]\left(\frac{x}{z}\right)^n = 1$$

let's call $(y/x)^n = s$ we have:

$$(1+s)\left(\frac{x}{z}\right)^n = 1 \Leftrightarrow \log(1+s) + n\log\left(\frac{x}{z}\right) = 0 \Leftrightarrow n = \frac{\log(1+s)}{\log z/x} \tag{10}$$

now:

$$\frac{y}{x} = s^{\frac{1}{n}} = s^{\frac{\log z/x}{\log(1+s)}}$$

extracting the base s logarithm and playing with logarithm properties we have:

$$\log_s y/x = \frac{\log z/x}{\log(1+s)} \Leftrightarrow \log_s(1+s) = \frac{\log z/x}{\log y/x}$$

so, recalling the values we chose for x, y, z:

$$log_s(1+s) = \frac{\log z/x}{\log y/x} \Leftrightarrow 1+s = s^{\frac{k}{m}} \Leftrightarrow (1+s)^m = s^k. \tag{11}$$

As we chose $k \leq 4$, by the Abel–Ruffini theorem we know we can find a closed form solution for (11). Now recalling equation (10) and substituting the hypotized values for x and z we derive (8).

In Table 1 we have a summary of the values of s according to the value of k and m.

For $(k, m) = (2, 1)$ we find again the class of solutions studied by Aiello, the other values give novel classes of solutions.

Using the values in the table we can easily find new triplets satisfying the Fermat's equation, for example for $a = 5$, $b = 6$, $m = 2$ and $k = 3$:

$$125^n + 180^n = 216^n$$

where

$$n = \frac{\log \frac{1}{3}\left(4+\sqrt[3]{\frac{47-3\sqrt{93}}{2}}+\sqrt[3]{\frac{47+3\sqrt{93}}{2}}\right)}{3\log 6/5} \approx 2.0965$$

As these exponents derive from an algebraic equation we'll call them Algebraic Derived Ratio of Irrationals, or ADRI numbers.

Are we satisfied with this result? Well, maybe some of you noticed that in Table 1 there's a strange row with $(k, m) = (5, 4)$, we can find that value analyzing the equation:

$$(1+s)^4 = s^5 \tag{12}$$

it is indeed divisible by $s^2 + s + 1$, so we can solve the remaining cubic equation to fill the last row in the table and find another series of ADRI numbers.

But we can say something more: the trinomial equation $s^u + ts^v = 1$ has been studied by Mellin and he found a series solution for it[8]:

$$s(u, v, t) = \frac{1}{u} \sum_{r=0} \frac{\Gamma\left(\frac{1+rv}{u}\right)}{\Gamma\left(\frac{1+rv}{u} + 1 - r\right) r!} (-1)^r t^r \tag{13}$$

so, if we consider closed-form enough a series expression for n we can prove the following:

Lemma 1 The equation (1), given positive integers x, y, z, has a solution for

$$n = \frac{\log\left[1 + \frac{1}{u} \sum_{r=0} \frac{\Gamma\left(\frac{1+r}{u}\right)}{\Gamma\left(\frac{1+r}{u}+1-r\right) r!}\right]}{\log z/x} \tag{14}$$

where

$$u = \frac{\log z/x}{\log y/x}$$

Proof We can prove the lemma starting from (11):

$$1 + s = s^{\frac{\log z/x}{\log y/x}} = s^u \tag{15}$$

We can recognize (15) as a trinomial equation with $v = 1$, $t = -1$ and u the ratio we already defined. So, substitution in (13) and then in (10) gives (14).

So, using this last lemma, we can state that the ADRI numbers consist in a (countable) infinity of classes of exponents, depending on the three natural numbers we want to push into the Fermat's equation.

1 Summary

It is known that the set of real exponents n for which $x^n + y^n = z^n$ for some positive integers x, y, z is dense in $\mathbb{R}$. But no closed form has been found for exponents greater than 2. In this paper we propose an algorithm to find new classes of exponents solving the Fermat's equation with values spanning to infinity. As a side result we also present an equation to find the generic exponent solving $x^n + y^n = z^n$ for every positive integer x, y and z.

2 References

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