RELAXED DISK PACKINGS

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ABSTRACT. The classic result about the optimal hexagonal packing of unit disks in the plane has recently been partially generalized by Edelsbrunner et al. to allow but penalize overlap for the case of lattice packings. We attempt to remove the restriction to lattice packings.

CONTENTS

1. INTRODUCTION

Recently, Edelsbrunner, Iglesias-Ham and Kurlin [EIK] considered a relaxed packing problem in which disks are allowed to overlap and the goal is to maximize the probability that a random point is contained in exactly one disk. They found that among all relaxed lattice packings of congruent disks in $\mathbb{R}^2$, a regular hexagonal packing with disks of a unique radius maximizes this probability. We attempt without success to remove the lattice hypothesis by generalizing Thomas Hales's proof [H1, H2] of Thue's Theorem, which is based on an idea of Rogers [R].

The key to Hales's proof is to partition the plane into three regions and show that the density of disks inside each region is no more than the hexagonal packing density, but when we allow disks to overlap, the density of disks inside one of the regions is larger than the relaxed hexagonal packing.

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2. THUE'S THEOREM

Thue's Theorem [T, 1892] says that the standard hexagonal packing is the densest packing of unit disks in the plane. We present an elegant proof by Hales [H1, H2] based on an idea of Rogers [R]. First we introduce lattices and weights.

Two-dimensional Lattices 2.1. A two-dimensional lattice L is defined by two linearly independent vectors, $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$ and consists of all integer combinations of these vectors:

$$L(\mathbf{v_1}, \mathbf{v_2}) = \{k_1\mathbf{v_1} + k_2\mathbf{v_2} : k_1, k_2 \in \mathbb{Z}\}$$

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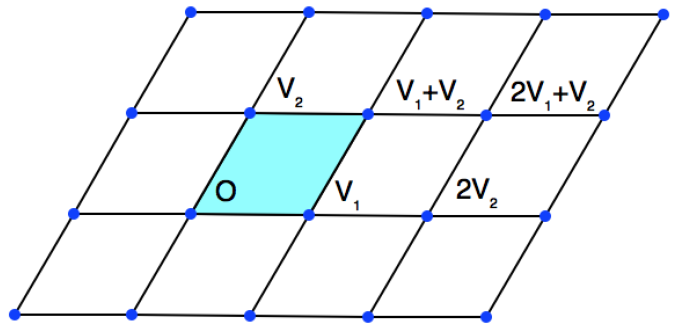

FIGURE 1. Two-dimensional lattice and its fundamental parallelogram

The parallelogram consisting of points

$$\theta_1\mathbf{v_1} + \theta_2\mathbf{v_2}\,(0 \le \theta_1 < 1,\, 0 \le \theta_2 < 1)$$

is a fundamental parallelogram. Figure 1 shows a two-dimensional lattice and the parallelogram determined by the $\mathbf{v_1}, \mathbf{v_2}$.

Weights 2.2. For a given packing of congruent disks $\mathcal{P}$ in $\mathbb{R}^2$, we use f_n to denote the fraction of space covered by exactly n disks and w_n to denote the weight assigned to that space, where $n = 0, 1, \ldots, n$. For example, the main result of Edelsbrunner et al. [EIK] finds the lattice packing $\mathcal{P}$ which maximizes $\sum_{i=0}^{n} w_i f_i(P)$ where $w_i = 0$ when $i \neq 1$ and $w_i = 1$ when $i = 1$. These are the weights that we use in our paper.

Lemma 2.3. *If $\mathcal{P}$ maximizes $\sum_{i=0}^{n} w_i f_i(P)$, then for every $a > 0$ and $b \in \mathbb{R}$, $\mathcal{P}$ also maximizes $\sum_{i=0}^{n} (aw_i + b) f_i(P)$.*

Proof. Let A be the set of values $\sum_{i=0}^{n}(w_i) f_i(P)$ where P is a packing. Then for $a > 0, b \in \mathbb{R}$, $aA + b$ is the set of values $\sum_{i=0}^{n}(aw_i + b) f_i(P)$, so that

$$\sup\,(aA + b) = a \sup A + b = \sum_{i=0}^{n}(aw_i + b) f_i(\mathcal{P}).$$

□

Lemma 2.4. *Suppose that $\mathcal{P}$ is a collection of disks of radius r. Then there exists a collection of unit disks such that $\mathcal{P}'$ such that $f_n(P) = f_n(P')$ for $n = 0, 1, \ldots, n$.*

Proof. The transformation $f(x, y) = (x/r,\, y/r)$ takes $\mathcal{P}$ to such a $\mathcal{P}'$ and preserves ratios of areas. □

Hales's proof 2.5. In 1892, Axel Thue showed that the regular hexagonal packing with density $\pi/\sqrt{12}$ is the highest-density packing of unit disks in the plane. Later, Hales gave an elementary proof of Thue's theorem in [H1, H2]. In his proof, Hales places a circle of radius $2/\sqrt{3}$ about

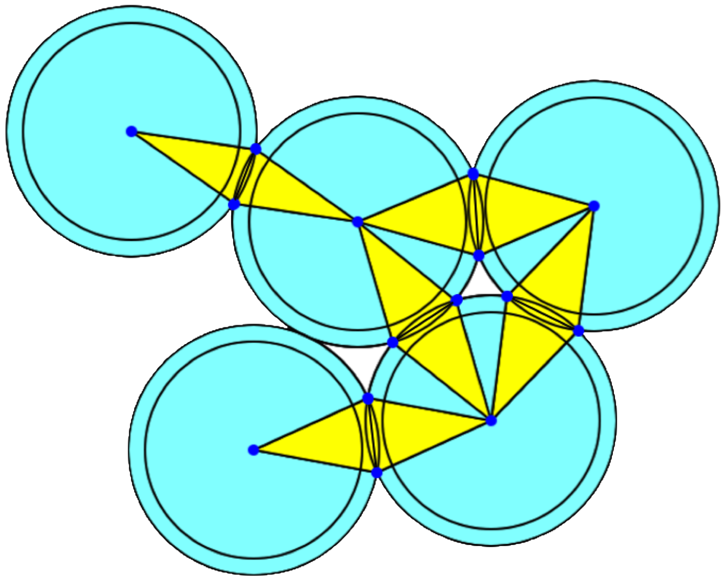

FIGURE 2. Partition the plane into three regions

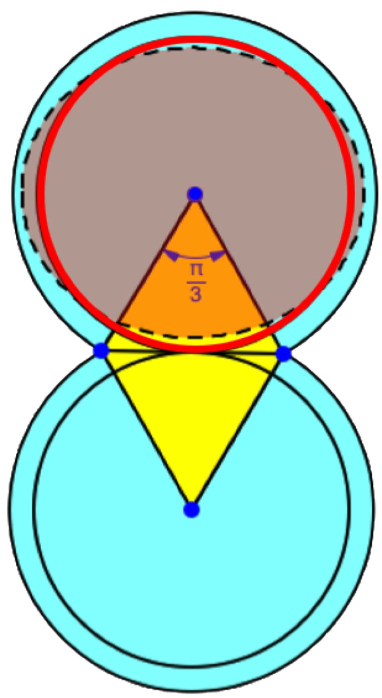

FIGURE 3. Linear Transformation

each center. This radius is exactly the side length of the Voronoi cells of the regular hexagonal lattice associated to the regular hexagonal packing. Whenever two of the larger disks intersect, two isosceles triangle are formed by taking the two intersection points and each of the two centers as vertices. This gives a partition of the plan into three parts: the region outside all of the larger circles, the region inside the larger circles but outside of the all triangles, and the region contained in the isosceles triangles as shown in Figure 2. The density of the first two regions is clearly less than $2/\sqrt{3}$. By choosing the radius of the larger circle to be $2/\sqrt{3}$, Hales ensures that the isosceles triangles will never overlap. This means that an upper bound on the density of individual triangles is an upper bound on the density of the region. Hales then uses a linear transformation to show that the density of an individual triangle is no more than the area of a unit disk divided by a hexagon of side length $2/\sqrt{3}$, i.e. no more than $\pi/\sqrt{12}$ as shown in Figure 3 . Note that this ideal case is actually achieved; the lattice associated to Thue's packing has Voronoi cells which are regular hexagons of side length $2/\sqrt{3}$ circumscribing the disks of radius 1.

3. Attempt to use Hales's Proof for Relaxed Disk Packings

Section 3 attempts to apply Hales's argument [H1, H2] to prove that the relaxed hexagonal packing is the densest relaxed disk packing. First we need to find the radius of the large disk which we will use later to truncate our Voronoi cell.

Radius of the Large Disk 3.1. According to the work done by Edelsbrunner et al. [EIK], the optimal relaxed-disk *lattice* packing is achieved by packing disks of radius $(\sqrt{6}-\sqrt{2})/2$ which centers on the hexagonal lattice generated by the vectors $\mathbf{v_1} = \langle 1, 0\rangle$ and $\mathbf{v_2} = \langle 1/2, \sqrt{3}/2\rangle$. For this optimal relaxed-disk lattice packing, the probability that a point lies inside exactly one disk is approximately 0.928.

In Hales's proof of Thue's Theorem, the radius of the larger circles is the side length of the regular hexagons forming the Voronoi cells of the optimal hexagonal packing. In order to emulate Hales's proof, we first find the side length of the Voronoi cells of the hexagonal relaxed packing which we conjectured to be optimal.

Figure 4 shows the conjectured optimal packing. since $\mathbf{v_1}$ and $\mathbf{v_2}$ each have length 1, ABC is an equilateral triangle with side length 1, so the height AE is $\sqrt{3}/2$. By Johnson's Theorem, we know that the point D is the centroid, so that

$$AD = \frac{2}{3}AE = \frac{\sqrt{3}}{3}$$

Therefore the radius of the Voronoi cells is $R = \sqrt{3}/3$. At the same time, we want to rescale the packing so that the disks have radius 1 instead of $(\sqrt{6}-\sqrt{2})/2$. After performing this rescaling, we have

$$R = \frac{\sqrt{3}}{3} \div \frac{(\sqrt{6}-\sqrt{2})}{2} = \frac{\sqrt{6}+3\sqrt{2}}{6}.$$

This shows that when attempting to adapt Hales's proof of Thue's Theorem to relaxed disk packings, we should attempt to partition the plane by playing larger disks of radius $(\sqrt{6}+3\sqrt{2})/6$ around the original disks.

Lemma 3.2. *If an optimal relaxed disk packing exists, then it must have overlapping disks.*

Proof. It suffices to show that the optimal non-overlapping packing is not the optimal relaxed packing.

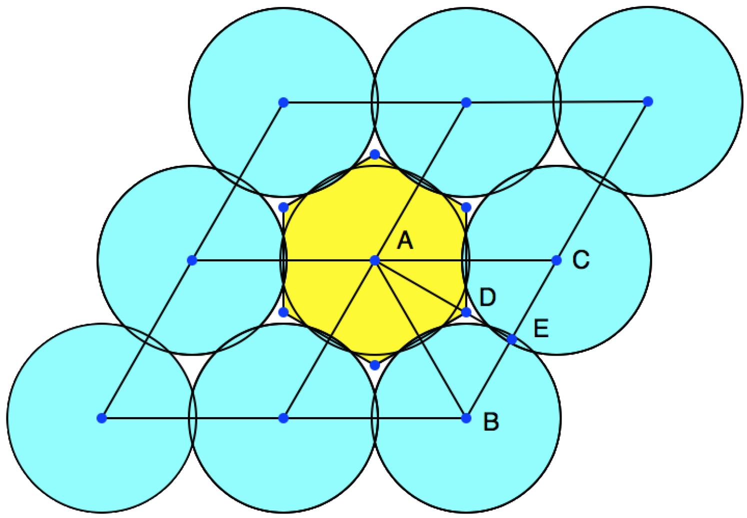

FIGURE 4. Radius of the large disk

Let $\mathcal{P}(R)$ be the packing in the hexagonal lattice generated by $\mathbf{v_1}$ and $\mathbf{v_2}$ of disks of radius R. Then $\mathcal{P}(1)$ is Thue's hexagonal packing. We have that

$$\frac{df_0}{dR}(1) < 0, \frac{df_1}{dR}(1) > 0 \text{ and } \frac{df_2}{dR}(1) = 0.$$

Therefore by making R slightly larger, the probability that a point will lie in exactly one disk will increase. This shows that $\mathcal{P}(1)$ is not the optimal relaxed packing, so that no non-overlapping packing can be optimal. □

Applying Hales's Proof 3.3. Let V be the set of centers of a collection of unit disks in $\mathbb{R}^2$. Take the Voronoi cell around each disk and truncate it by intersecting it with a large disk of radius $R = \left(\sqrt{6} + 3\sqrt{2}\right)/6$. See Figure 6. This gives a partition of the plane into three regions: regions outside all of the larger circles, the isosceles triangles, and the part inside the larger circles but outside all triangles. The region outside all of the larger disks has density

$$0 < 0.928\ldots.$$

and the region that is inside the large circles but outside of the triangles has density less than or equal to

$$\frac{\text{Area of small disks}}{\text{Area of larger disks}} = \frac{\pi}{\pi R^2} = 6 - 3\sqrt{3} = 0.803\ldots < 0.928\ldots.$$

Let x be the distance between the center of the two disks. The area of the large isosceles triangle ABC is

$$\frac{1}{2}AH \cdot BC = AH \cdot BH = \frac{x}{2} \cdot \sqrt{R^2 - \frac{x^2}{4}}$$

The area of the small isosceles triangle ADE is

$$\frac{1}{2}AH \cdot DE = AH \cdot DH = \frac{x}{2} \cdot \sqrt{1 - \frac{x^2}{4}}.$$

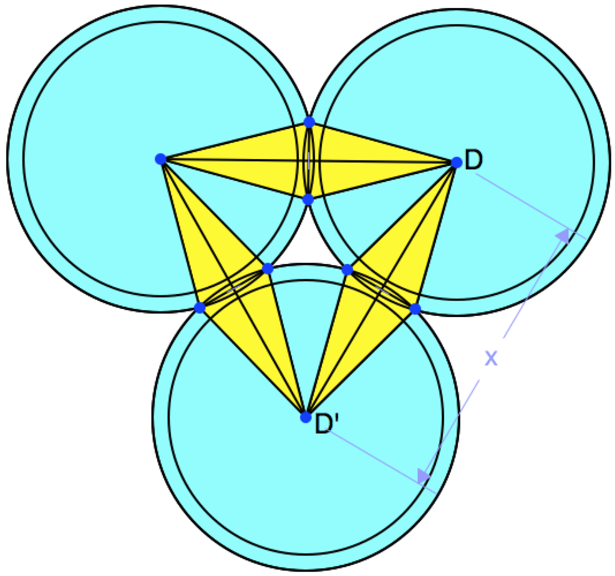

FIGURE 5. Disks must overlap

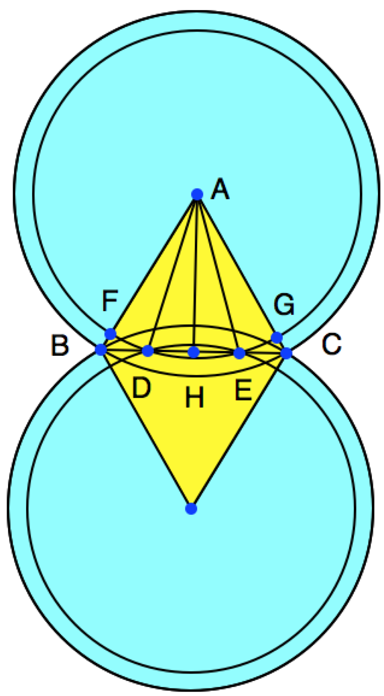

FIGURE 6. Overlaps of two disks

Since $\cos(\angle BAH) = AH/AB = x/(2R)$, we have

$$\cos(\angle BAC) = 2\cos^2(\angle BAH) - 1 = \frac{x^2}{2R^2} - 1.$$

This gives us

$$\angle BAC = \arccos(\frac{x^2}{2R^2} - 1)$$

The area of the large sector AFG is

$$\frac{1}{2}r^2 \cdot \angle BAC = \frac{1}{2} \cdot \arccos(\frac{x^2}{2R^2} - 1)$$

Since the $\cos(\angle DAH) = AH/AD = x/2$, we have

$$\cos(\angle DAE) = 2\cos^2(\angle DAH) - 1 = 2 \cdot \frac{x^2}{4} - 1 = \frac{x^2}{2} - 1.$$

This gives us

$$\angle DAE = \arccos(\frac{x^2}{2} - 1)$$

The area of the small sector ADE is

$$\frac{1}{2} \cdot r^2 \angle DAE = \frac{1}{2} \cdot \arccos(\frac{x^2}{2} - 1)$$

The area of the overlaps of two unit disks is

$$2\cdot(\text{sector AGH}-\text{triangle AGH}) = 2\cdot(\frac{1}{2}\cdot\arccos(\frac{x^2}{2}-1)-\frac{x}{2}\cdot\sqrt{1-\frac{x^2}{4}}) = \arccos(\frac{x^2}{2}-1)-x\cdot\sqrt{1-\frac{x^2}{4}}$$

The area of the weighted sector AFG is

$$\frac{1}{2} \cdot \arccos(\frac{x^2}{2R^2} - 1) - \arccos(\frac{x^2}{2} - 1) + x \cdot \sqrt{1 - \frac{x^2}{4}}$$

Thus, the density inside the triangle ABC is

$$\frac{\text{weighted area of the sector AFG}}{\text{area of the triangle ABC}} = \frac{\frac{1}{2} \cdot \arccos(\frac{x^2}{2R^2} - 1) - \arccos(\frac{x^2}{2} - 1) + x \cdot \sqrt{1 - \frac{x^2}{4}}}{\frac{x}{2} \cdot \sqrt{R^2 - \frac{x^2}{4}}}$$

By using $R = (\sqrt{6} + 3\sqrt{2})/6$, we get

$$\frac{\text{weighted area of the sector AFG}}{\text{area of the triangle ABC}} = \frac{2x\sqrt{4-x^2} - 4\arccos(\frac{x^2}{2} - 1) + 2\arccos(-\frac{3}{2}(\sqrt{3}-2)x^2 - 1)}{x\sqrt{\frac{4}{3}(2+\sqrt{3}) - x^2}}$$

Mathematica shows that this is achieves a maximum value of $0.934\ldots$ when $x = 1.973\ldots$, which is larger than $0.928\ldots$. See Figure 7.

This does not disprove the conjecture. Although the density of the triangle exceeds 0.928 when $x = 1.973\ldots$, at this time the density of the region that lies outside the triangles but inside the Voronoi cell of the each disk is relatively small. This brings down the density of the packing to less than $0.928\ldots$. See Figure 8.

The salient feature of Hales's proof of Thue's theorem is that the density of the triangle achieves its maximum exactly when there is no empty space between the triangles and the triangles tile the plane. Unfortunately, this is not the case we encounter.

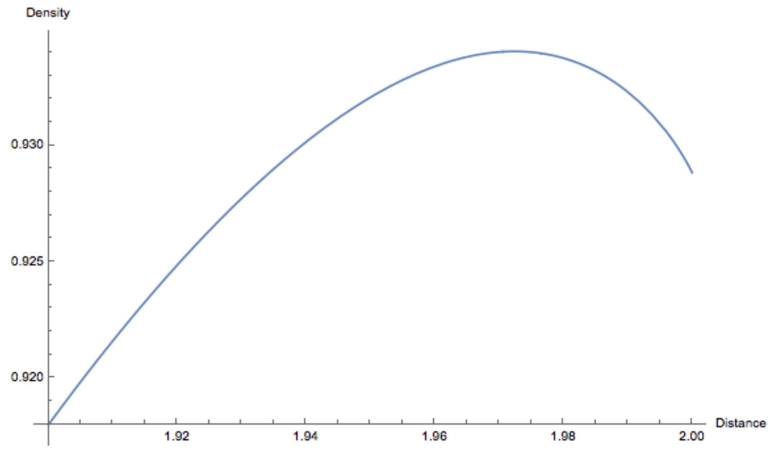

FIGURE 7. The density of the triangle with respect to distance x

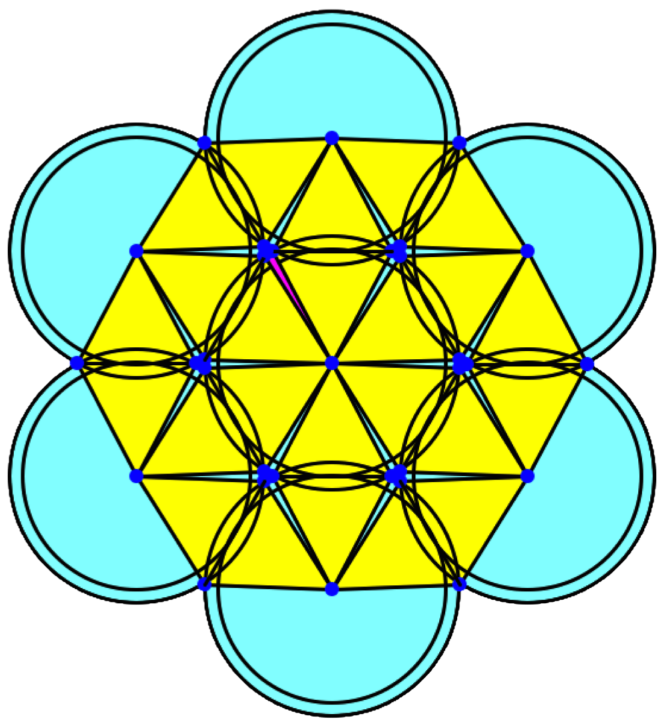

FIGURE 8. The density of cyan area achieves maximum, but the density of the pink area is relatively small.

We have not even shown that an optimal relaxed disk packing exists. The existence of an optimal disk packing follows from Thue's Theorem; it can also be proved by showing that because only a finite number of non-overlapping disks can fit in a given square, an optimal packing exists for squares of arbitrary size. It is not immediate that only a finite number of overlapping disks can fit in a given square without decreasing the density.

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