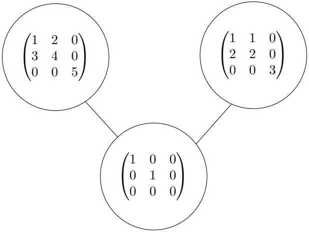


Higher Distance Commuting Varieties

Ralph Morrison (Williams College, visitor at MPI Leipzig)

Joint work with Madeleine Elyze and Alexander Guterman

Nonlinear Algebra Seminar Online
March 17, 2020

Commuting Matrices

Let $\mathrm{Mat}_n(k)$ be the set of all $n \times n$ matrices with entries in a field k.

Commuting Matrices

Let $\mathrm{Mat}_n(k)$ be the set of all $n \times n$ matrices with entries in a field k.

We say $A, B \in \mathrm{Mat}_n(k)$ **commute** if

$$AB = BA,$$

and denote this by $A \leftrightarrow B$.

Commuting Matrices

Let $\mathrm{Mat}_n(k)$ be the set of all $n \times n$ matrices with entries in a field k.

We say $A, B \in \mathrm{Mat}_n(k)$ **commute** if

$$AB = BA,$$

and denote this by $A \leftrightarrow B$.

So if

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $A \leftrightarrow C$ and $B \leftrightarrow C$, but $A \not\leftrightarrow B$.

Commuting Matrices

Let $\mathrm{Mat}_n(k)$ be the set of all $n \times n$ matrices with entries in a field k.

We say $A, B \in \mathrm{Mat}_n(k)$ **commute** if

$$AB = BA,$$

and denote this by $A \leftrightarrow B$.

So if

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $A \leftrightarrow C$ and $B \leftrightarrow C$, but $A \not\leftrightarrow B$.

If $S = \lambda I$ for some $\lambda \in k$, we call S a **scalar** matrix.

Commuting Matrices

Let $\mathrm{Mat}_n(k)$ be the set of all $n \times n$ matrices with entries in a field k.

We say $A, B \in \mathrm{Mat}_n(k)$ **commute** if

$$AB = BA,$$

and denote this by $A \leftrightarrow B$.

So if

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $A \leftrightarrow C$ and $B \leftrightarrow C$, but $A \not\leftrightarrow B$.

If $S = \lambda I$ for some $\lambda \in k$, we call S a **scalar** matrix.

If S is scalar, then for all $A \in \mathrm{Mat}_n(k)$ we have $S \leftrightarrow A$.

The Commuting Graph

Define the **commuting graph** $\Gamma(n, k)$ as follows:

The Commuting Graph

Define the **commuting graph** $\Gamma(n, k)$ as follows:

- The vertices are the **nonscalar** elements of $\mathrm{Mat}_n(k)$.

The Commuting Graph

Define the **commuting graph** $\Gamma(n, k)$ as follows:

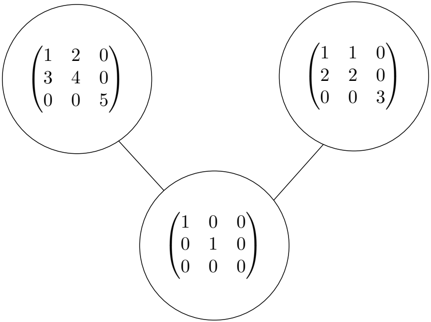
- The vertices are the **nonscalar** elements of $\mathrm{Mat}_n(k)$.
- The edges connect pairs of matrices that commute.

The Commuting Graph

Define the **commuting graph** $\Gamma(n, k)$ as follows:

- The vertices are the **nonscalar** elements of $\mathrm{Mat}_n(k)$.
- The edges connect pairs of matrices that commute.

Inside the graph $\Gamma(3, \mathbb{R})$, we would see the following three vertices and two edges:

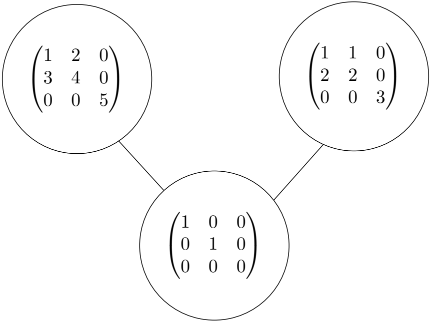


The Commuting Graph

Define the **commuting graph** $\Gamma(n, k)$ as follows:

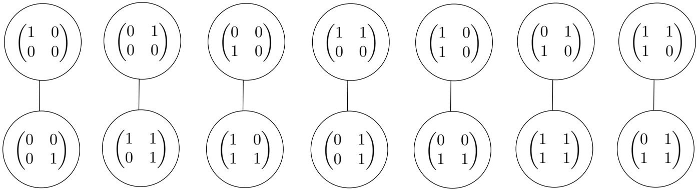
- The vertices are the **nonscalar** elements of $\mathrm{Mat}_n(k)$.
- The edges connect pairs of matrices that commute.

Inside the graph $\Gamma(3, \mathbb{R})$, we would see the following three vertices and two edges:

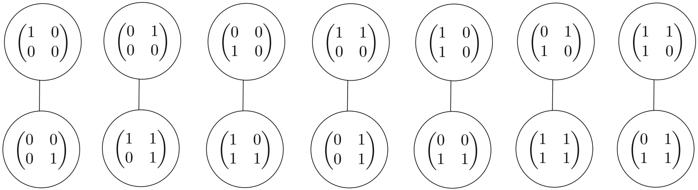


Is the commuting graph **connected**? If so, what is its **diameter**?

Example: the 2×2 commuting graph over $\mathbb{Z}/2\mathbb{Z}$

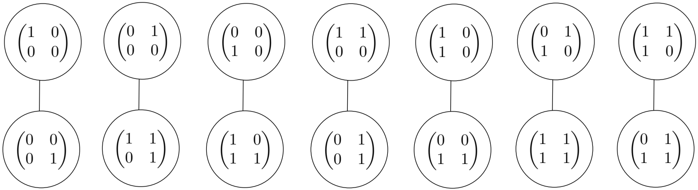


Example: the 2×2 commuting graph over $\mathbb{Z}/2\mathbb{Z}$



$\Gamma(2, k)$ is disconnected for any field k (Akbari-Raja, 2006).

Example: the 2×2 commuting graph over $\mathbb{Z}/2\mathbb{Z}$

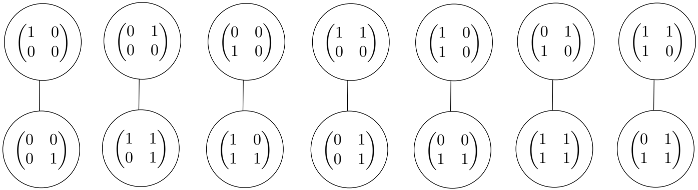


$\Gamma(2, k)$ is disconnected for any field k (Akbari-Raja, 2006).

For $A \in \mathrm{Mat}_2(k)$,

$$A \text{ is nonscalar} \Leftrightarrow \mathrm{char}(A) = \min(A)$$

Example: the 2×2 commuting graph over $\mathbb{Z}/2\mathbb{Z}$

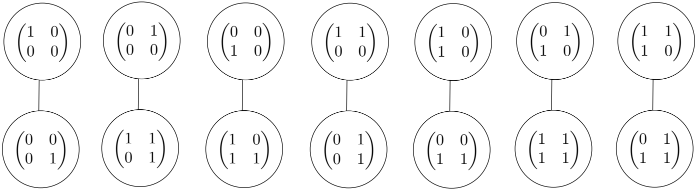


$\Gamma(2, k)$ is disconnected for any field k (Akbari-Raja, 2006).

For $A \in \mathrm{Mat}_2(k)$,

$$\begin{aligned} A \text{ is nonscalar} &\Leftrightarrow \mathrm{char}(A) = \min(A) \\ &\Leftrightarrow A \text{ only commutes with polynomials in } A \end{aligned}$$

Example: the 2×2 commuting graph over $\mathbb{Z}/2\mathbb{Z}$



$\Gamma(2, k)$ is disconnected for any field k (Akbari-Raja, 2006).

For $A \in \mathrm{Mat}_2(k)$,

$$\begin{aligned} A \text{ is nonscalar} &\Leftrightarrow \mathrm{char}(A) = \min(A) \\ &\Leftrightarrow A \text{ only commutes with polynomials in } A \end{aligned}$$

If $A \leftrightarrow C \leftrightarrow B$, then $A = p(C)$ and $B = q(C)$, so $A \leftrightarrow B$.

Connectivity when $n \geq 3$

Theorem (Akbari-Mohammadian-Radjavi-Raja, 2006)

If $n \geq 3$ *then* $\Gamma(n, \mathbb{C})$ *is connected, and* $diam(\Gamma(n, \mathbb{C})) = 4$.

Connectivity when $n \geq 3$

Theorem (Akbari-Mohammadian-Radjavi-Raja, 2006)

If $n \geq 3$ then $\Gamma(n, \mathbb{C})$ is connected, and diam$(\Gamma(n, \mathbb{C})) = 4$.

Proof hint: Given A and B, show that there exist rank 1 matrices C, D, E with

$$A \leftrightarrow C \leftrightarrow D \leftrightarrow E \leftrightarrow B$$

Connectivity when $n \geq 3$

Theorem (Akbari-Mohammadian-Radjavi-Raja, 2006)

If $n \geq 3$ then $\Gamma(n, \mathbb{C})$ is connected, and diam$(\Gamma(n, \mathbb{C})) = 4$.

Proof hint: Given A and B, show that there exist rank 1 matrices C, D, E with

$$A \leftrightarrow C \leftrightarrow D \leftrightarrow E \leftrightarrow B$$

- The same theorem holds if $k = \mathbb{R}$ (Shitov, 2018).

Connectivity when $n \geq 3$

Theorem (Akbari-Mohammadian-Radjavi-Raja, 2006)

If $n \geq 3$ then $\Gamma(n, \mathbb{C})$ is connected, and diam$(\Gamma(n, \mathbb{C})) = 4$.

Proof hint: Given A and B, show that there exist rank 1 matrices C, D, E with

$$A \leftrightarrow C \leftrightarrow D \leftrightarrow E \leftrightarrow B$$

- The same theorem holds if $k = \mathbb{R}$ (Shitov, 2018).
- For any n, $\Gamma(n, \mathbb{Q})$ is disconnected (Akbari-Bidkhori-Mohammadian, 2008).

Commuting distance

Let $n \geq 3$, and $A, B \in \mathrm{Mat}_n(k)$ be nonscalar. Their **commuting distance** $d(A, B)$ is the distance between A and B on $\Gamma(n, k)$.

Commuting distance

Let $n \geq 3$, and $A, B \in \mathrm{Mat}_n(k)$ be nonscalar. Their **commuting distance** $d(A, B)$ is the distance between A and B on $\Gamma(n, k)$.

- $d(A, B) = 0$ if and only if $A = B$

Commuting distance

Let $n \geq 3$, and $A, B \in \mathrm{Mat}_n(k)$ be nonscalar. Their **commuting distance** $d(A, B)$ is the distance between A and B on $\Gamma(n, k)$.

- $d(A, B) = 0$ if and only if $A = B$
- $d(A, B) = 1$ if and only if $A \neq B$ and $A \leftrightarrow B$

Commuting distance

Let $n \geq 3$, and $A, B \in \mathrm{Mat}_n(k)$ be nonscalar. Their **commuting distance** $d(A, B)$ is the distance between A and B on $\Gamma(n, k)$.

- $d(A, B) = 0$ if and only if $A = B$
- $d(A, B) = 1$ if and only if $A \neq B$ and $A \leftrightarrow B$
- $d(A, B) = d$ if and only if d is the smallest natural number such that there exist $d - 1$ nonscalar matrices $C_1, \ldots, C_{d-1}$ with

$$A \leftrightarrow C_1 \leftrightarrow \cdots \leftrightarrow C_{d-1} \leftrightarrow B$$

Commuting distance

Let $n \geq 3$, and $A, B \in \mathrm{Mat}_n(k)$ be nonscalar. Their **commuting distance** $d(A, B)$ is the distance between A and B on $\Gamma(n, k)$.

- $d(A, B) = 0$ if and only if $A = B$
- $d(A, B) = 1$ if and only if $A \neq B$ and $A \leftrightarrow B$
- $d(A, B) = d$ if and only if d is the smallest natural number such that there exist $d - 1$ nonscalar matrices $C_1, \ldots, C_{d-1}$ with
$$A \leftrightarrow C_1 \leftrightarrow \cdots \leftrightarrow C_{d-1} \leftrightarrow B$$

So

$$d\left(\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right) = 2$$

Commuting distance

Let $n \geq 3$, and $A, B \in \mathrm{Mat}_n(k)$ be nonscalar. Their **commuting distance** $d(A, B)$ is the distance between A and B on $\Gamma(n, k)$.

- $d(A, B) = 0$ if and only if $A = B$
- $d(A, B) = 1$ if and only if $A \neq B$ and $A \leftrightarrow B$
- $d(A, B) = d$ if and only if d is the smallest natural number such that there exist $d - 1$ nonscalar matrices $C_1, \ldots, C_{d-1}$ with

$$A \leftrightarrow C_1 \leftrightarrow \cdots \leftrightarrow C_{d-1} \leftrightarrow B$$

So

$$d\left(\begin{pmatrix}1 & 2 & 0\\ 3 & 4 & 0\\ 0 & 0 & 5\end{pmatrix}, \begin{pmatrix}1 & 1 & 0\\ 2 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}\right) = 2$$

We sometimes extend this definition to include scalar matrices by declaring $d(A, B) = 1$ when A or B is scalar (and $A \neq B$).

The distance d commuting set

Let $d \geq 0$. The **distance-d commuting set** $\mathcal{C}_n^d$ is set of pairs of $n \times n$ matrices with commuting distance at most d:

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2.$$

The distance d commuting set

Let $d \geq 0$. The **distance-d commuting set** $\mathcal{C}_n^d$ is set of pairs of $n \times n$ matrices with commuting distance at most d:

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2.$$

For example,

$$\mathcal{C}_n^1 = \{(A, B) \mid d(A, B) \leq 1\}$$

The distance d commuting set

Let $d \geq 0$. The **distance-d commuting set** $\mathcal{C}_n^d$ is set of pairs of $n \times n$ matrices with commuting distance at most d:

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2.$$

For example,

$$\mathcal{C}_n^1 = \{(A, B) \mid d(A, B) \leq 1\} = \{(A, B) \mid A \leftrightarrow B\}.$$

The distance d commuting set

Let $d \geq 0$. The **distance-d commuting set** $\mathcal{C}_n^d$ is set of pairs of $n \times n$ matrices with commuting distance at most d:

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2.$$

For example,

$$\mathcal{C}_n^1 = \{(A, B) \mid d(A, B) \leq 1\} = \{(A, B) \mid A \leftrightarrow B\}.$$

Theorem (Elyze-Guterman-M., 2018)

If k is $\mathbb{C}$ or $\mathbb{R}$, then set $\mathcal{C}_n^d$ is an **affine variety** *inside the $2n^2$-dimensional space $Mat_n(k)^2$.*

The distance d commuting set

Let $d \geq 0$. The **distance-d commuting set** $\mathcal{C}_n^d$ is set of pairs of $n \times n$ matrices with commuting distance at most d:

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2.$$

For example,

$$\mathcal{C}_n^1 = \{(A, B) \mid d(A, B) \leq 1\} = \{(A, B) \mid A \leftrightarrow B\}.$$

Theorem (Elyze-Guterman-M., 2018)

If k is $\mathbb{C}$ or $\mathbb{R}$, then set $\mathcal{C}_n^d$ is an **affine variety** *inside the $2n^2$-dimensional space $Mat_n(k)^2$.*
The same holds for any field k if $d \leq 2$.

Algebraic geometry

Algebraic geometry is the study of solutions to polynomial equations over a field k.

Algebraic geometry

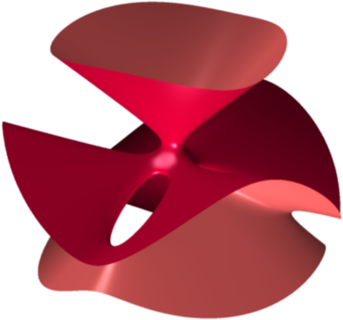
Algebraic geometry is the study of solutions to polynomial equations over a field k.

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$

Algebraic geometry

Algebraic geometry is the study of solutions to polynomial equations over a field k.

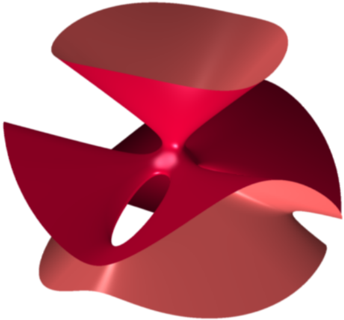
$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$



Algebraic geometry

Algebraic geometry is the study of solutions to polynomial equations over a field k.

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$



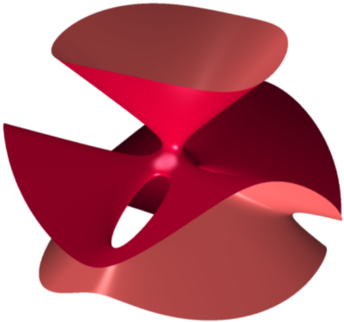
For $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, define

$$\mathbf{V}(f_1, \dots, f_s) := \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0 \text{ for } 1 \leq i \leq s\}.$$

Algebraic geometry

Algebraic geometry is the study of solutions to polynomial equations over a field k.

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$



For $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, define

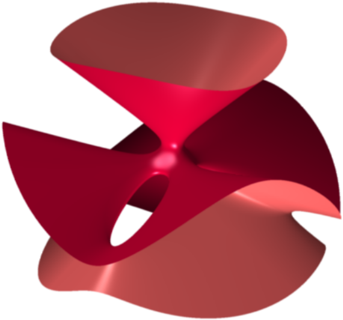
$$\mathbf{V}(f_1, \ldots, f_s) := \{(a_1, \ldots, a_n) \in k^n \mid f_i(a_1, \ldots, a_n) = 0 \text{ for } 1 \leq i \leq s\}.$$

This is the **(affine) variety** defined by $f_1, \ldots, f_s$.

Algebraic geometry

Algebraic geometry is the study of solutions to polynomial equations over a field k.

$$x^3 + y^3 + z^3 + 1 - (x + y + z + 1)^3 = 0$$



For $f_1, \dots, f_s \in k[x_1, \dots, x_n]$, define

$$\mathbf{V}(f_1, \dots, f_s) := \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0 \text{ for } 1 \leq i \leq s\}.$$

This is the **(affine) variety** defined by $f_1, \dots, f_s$.

Varieties in Linear Algebra: Singular Matrices

Consider the set of all 2×2 matrices over $\mathbb{R}$:

$$\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Varieties in Linear Algebra: Singular Matrices

Consider the set of all 2×2 matrices over $\mathbb{R}$:

$$\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Thinking of $\mathrm{Mat}_2(\mathbb{R})$ as $\mathbb{R}^4$, we can let $f \in \mathbb{R}[w, x, y, z]$ and look at

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid f(a, b, c, d) = 0 \right\} \subset \mathrm{Mat}_2(\mathbb{R}).$$

Varieties in Linear Algebra: Singular Matrices

Consider the set of all 2×2 matrices over $\mathbb{R}$:

$$\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Thinking of $\mathrm{Mat}_2(\mathbb{R})$ as $\mathbb{R}^4$, we can let $f \in \mathbb{R}[w, x, y, z]$ and look at

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid f(a, b, c, d) = 0 \right\} \subset \mathrm{Mat}_2(\mathbb{R}).$$

For example, if $f = wz - xy$, then

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 0 \right\}.$$

Varieties in Linear Algebra: Singular Matrices

Consider the set of all 2×2 matrices over $\mathbb{R}$:

$$\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Thinking of $\mathrm{Mat}_2(\mathbb{R})$ as $\mathbb{R}^4$, we can let $f \in \mathbb{R}[w, x, y, z]$ and look at

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid f(a, b, c, d) = 0 \right\} \subset \mathrm{Mat}_2(\mathbb{R}).$$

For example, if $f = wz - xy$, then

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 0 \right\}.$$

That is, $\mathbf{V}(f)$ is the set of all singular matrices.

Varieties in Linear Algebra: Singular Matrices

Consider the set of all 2×2 matrices over $\mathbb{R}$:

$$\mathrm{Mat}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

Thinking of $\mathrm{Mat}_2(\mathbb{R})$ as $\mathbb{R}^4$, we can let $f \in \mathbb{R}[w, x, y, z]$ and look at

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid f(a, b, c, d) = 0 \right\} \subset \mathrm{Mat}_2(\mathbb{R}).$$

For example, if $f = wz - xy$, then

$$\mathbf{V}(f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 0 \right\}.$$

That is, $\mathbf{V}(f)$ is the set of all singular matrices.
(This works for any n, choosing f to be the $n \times n$ determinant polynomial.)

Varieties in Linear Algebra: Matrices of Rank at most r

Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

Varieties in Linear Algebra: Matrices of Rank at most r

Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

A matrix has rank at most r if and only if all $(r+1) \times (r+1)$ minors are 0.

Varieties in Linear Algebra: Matrices of Rank at most r

Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

A matrix has rank at most r if and only if all $(r+1) \times (r+1)$ minors are 0.

So $\mathrm{rank}(A) \leq 1$ if and only if

$$a_{11}a_{22} - a_{12}a_{21} = 0, \quad a_{11}a_{23} - a_{13}a_{21} = 0, \quad a_{22}a_{23} - a_{13}a_{22} = 0.$$

Varieties in Linear Algebra: Matrices of Rank at most r

Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

A matrix has rank at most r if and only if all $(r+1) \times (r+1)$ minors are 0.

So $\mathrm{rank}(A) \leq 1$ if and only if

$$a_{11}a_{22} - a_{12}a_{21} = 0, \quad a_{11}a_{23} - a_{13}a_{21} = 0, \quad a_{22}a_{23} - a_{13}a_{22} = 0.$$

Let $f_1, f_2, f_3 \in \mathbb{R}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$ be defined by

$$f_1 = x_{11}x_{22} - x_{12}x_{21}, \quad f_2 = x_{11}x_{23} - x_{13}x_{21}, \quad f_3 = x_{22}x_{23} - x_{13}x_{22}.$$

Varieties in Linear Algebra: Matrices of Rank at most r

Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

A matrix has rank at most r if and only if all $(r+1) \times (r+1)$ minors are 0.

So $\mathrm{rank}(A) \leq 1$ if and only if

$$a_{11}a_{22} - a_{12}a_{21} = 0, \quad a_{11}a_{23} - a_{13}a_{21} = 0, \quad a_{22}a_{23} - a_{13}a_{22} = 0.$$

Let $f_1, f_2, f_3 \in \mathbb{R}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$ be defined by

$$f_1 = x_{11}x_{22} - x_{12}x_{21}, \quad f_2 = x_{11}x_{23} - x_{13}x_{21}, \quad f_3 = x_{22}x_{23} - x_{13}x_{22}.$$

Then $\mathbf{V}(f_1, f_2, f_3) = \{A \,|\, \mathrm{rank}(A) \leq 1\} \subset \mathrm{Mat}_{2\times 3}(\mathbb{R})$.

Varieties in Linear Algebra: Matrices of Rank at most r

Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}.$$

A matrix has rank at most r if and only if all $(r+1) \times (r+1)$ minors are 0.

So $\operatorname{rank}(A) \leq 1$ if and only if

$$a_{11}a_{22} - a_{12}a_{21} = 0, \quad a_{11}a_{23} - a_{13}a_{21} = 0, \quad a_{22}a_{23} - a_{13}a_{22} = 0.$$

Let $f_1, f_2, f_3 \in \mathbb{R}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}]$ be defined by

$$f_1 = x_{11}x_{22} - x_{12}x_{21}, \quad f_2 = x_{11}x_{23} - x_{13}x_{21}, \quad f_3 = x_{22}x_{23} - x_{13}x_{22}.$$

Then $\mathbf{V}(f_1, f_2, f_3) = \{A \,|\, \operatorname{rank}(A) \leq 1\} \subset \mathrm{Mat}_{2\times 3}(\mathbb{R})$.

In general, the set of all $m \times n$ matrices with rank at most r is a variety.

Varieties in Linear Algebra: the Commuting Variety

Let

$$(A, B) = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \in \mathrm{Mat}_2(\mathbb{R})^2$$

Varieties in Linear Algebra: the Commuting Variety

Let

$$(A, B) = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \in \mathrm{Mat}_2(\mathbb{R})^2$$

Then $AB - BA = 0$ if and only if the following polynomials vanish when we plug in the entries of A and B:

$$\begin{aligned} f_1 =& x_{11}y_{11} + x_{12}y_{21} - x_{11}y_{11} - x_{21}y_{12} \\ f_2 =& x_{11}y_{12} + x_{12}y_{22} - x_{12}y_{11} - x_{22}y_{12} \\ f_3 =& x_{21}y_{11} + x_{22}y_{21} - x_{11}y_{21} - x_{21}y_{22} \\ f_4 =& x_{21}y_{12} + x_{22}y_{22} - x_{12}y_{21} - x_{22}y_{22}. \end{aligned}$$

Varieties in Linear Algebra: the Commuting Variety

Let

$$(A, B) = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \in \mathrm{Mat}_2(\mathbb{R})^2$$

Then $AB - BA = 0$ if and only if the following polynomials vanish when we plug in the entries of A and B:

$$\begin{aligned} f_1 =& x_{11}y_{11} + x_{12}y_{21} - x_{11}y_{11} - x_{21}y_{12} \\ f_2 =& x_{11}y_{12} + x_{12}y_{22} - x_{12}y_{11} - x_{22}y_{12} \\ f_3 =& x_{21}y_{11} + x_{22}y_{21} - x_{11}y_{21} - x_{21}y_{22} \\ f_4 =& x_{21}y_{12} + x_{22}y_{22} - x_{12}y_{21} - x_{22}y_{22}. \end{aligned}$$

So $\mathbf{V}(f_1, f_2, f_3, f_4) = \{(A, B) \,|\, A \leftrightarrow B\} \subset \mathrm{Mat}_2(\mathbb{R})^2$.

Varieties in Linear Algebra: the Commuting Variety

Let

$$(A, B) = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \in \mathrm{Mat}_2(\mathbb{R})^2$$

Then $AB - BA = 0$ if and only if the following polynomials vanish when we plug in the entries of A and B:

$$\begin{aligned} f_1 &= x_{11}y_{11} + x_{12}y_{21} - x_{11}y_{11} - x_{21}y_{12} \\ f_2 &= x_{11}y_{12} + x_{12}y_{22} - x_{12}y_{11} - x_{22}y_{12} \\ f_3 &= x_{21}y_{11} + x_{22}y_{21} - x_{11}y_{21} - x_{21}y_{22} \\ f_4 &= x_{21}y_{12} + x_{22}y_{22} - x_{12}y_{21} - x_{22}y_{22}. \end{aligned}$$

So $\mathbf{V}(f_1, f_2, f_3, f_4) = \{(A, B) \mid A \leftrightarrow B\} \subset \mathrm{Mat}_2(\mathbb{R})^2$.

This also works for $n > 2$. We call the set of commuting pairs of $n \times n$ matrices the $n \times n$ **commuting variety** $\mathcal{C}_n$.

Varieties in Linear Algebra: the Commuting Variety

Let

$$(A, B) = \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \in \mathrm{Mat}_2(\mathbb{R})^2$$

Then $AB - BA = 0$ if and only if the following polynomials vanish when we plug in the entries of A and B:

$$\begin{aligned} f_1 &= x_{11}y_{11} + x_{12}y_{21} - x_{11}y_{11} - x_{21}y_{12} \\ f_2 &= x_{11}y_{12} + x_{12}y_{22} - x_{12}y_{11} - x_{22}y_{12} \\ f_3 &= x_{21}y_{11} + x_{22}y_{21} - x_{11}y_{21} - x_{21}y_{22} \\ f_4 &= x_{21}y_{12} + x_{22}y_{22} - x_{12}y_{21} - x_{22}y_{22}. \end{aligned}$$

So $\mathbf{V}(f_1, f_2, f_3, f_4) = \{(A, B) \mid A \leftrightarrow B\} \subset \mathrm{Mat}_2(\mathbb{R})^2$.

This also works for $n > 2$. We call the set of commuting pairs of $n \times n$ matrices the $n \times n$ **commuting variety** $\mathcal{C}_n$.

It is **irreducible**, with **dimension** $n^2 + n$.

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	?	?	?	?
$\mathbb{R}$	?	?	?	?
k	?	?	?	?

A few easy cases

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

A few easy cases

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A, B) \mid A = B\}$

A few easy cases

$$\mathcal{C}_n^d := \{(A,B) \mid d(A,B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A,B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$

A few easy cases

$$\mathcal{C}_n^d := \{(A,B) \mid d(A,B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A,B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$
- $\mathcal{C}_n^1 = \{(A,B) \mid A \leftrightarrow B\}$

A few easy cases

$$\mathcal{C}_n^d := \{(A,B) \mid d(A,B) \le d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A,B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$
- $\mathcal{C}_n^1 = \{(A,B) \mid A \leftrightarrow B\} = \mathcal{C}_n$

A few easy cases

$$\mathcal{C}_n^d := \{(A,B) \mid d(A,B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A,B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$
- $\mathcal{C}_n^1 = \{(A,B) \mid A \leftrightarrow B\} = \mathcal{C}_n$

If k is $\mathbb{C}$ or $\mathbb{R}$ then $d(A,B) \leq 4$ for all A and B, so:

A few easy cases

$$\mathcal{C}_n^d := \{(A, B) \mid d(A, B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A, B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$
- $\mathcal{C}_n^1 = \{(A, B) \mid A \leftrightarrow B\} = \mathcal{C}_n$

If k is $\mathbb{C}$ or $\mathbb{R}$ then $d(A, B) \leq 4$ for all A and B, so:

- $\mathcal{C}_n^4 = \mathrm{Mat}_n(k)^2$

A few easy cases

$$\mathcal{C}_n^d := \{(A,B) \mid d(A,B) \leq d\} \subset \mathsf{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A,B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$
- $\mathcal{C}_n^1 = \{(A,B) \mid A \leftrightarrow B\} = \mathcal{C}_n$

If k is $\mathbb{C}$ or $\mathbb{R}$ then $d(A,B) \leq 4$ for all A and B, so:

- $\mathcal{C}_n^4 = \mathrm{Mat}_n(k)^2 = \mathbf{V}(0)$

A few easy cases

$$\mathcal{C}_n^d := \{(A,B) \mid d(A,B) \leq d\} \subset \mathrm{Mat}_n(k)^2$$

Over any field:

- $\mathcal{C}_n^0 = \{(A,B) \mid A = B\} = \mathbf{V}(x_{11} - y_{11}, x_{12} - y_{12}, \ldots, x_{nn} - y_{nn})$
- $\mathcal{C}_n^1 = \{(A,B) \mid A \leftrightarrow B\} = \mathcal{C}_n$

If k is $\mathbb{C}$ or $\mathbb{R}$ then $d(A,B) \leq 4$ for all A and B, so:

- $\mathcal{C}_n^4 = \mathrm{Mat}_n(k)^2 = \mathbf{V}(0)$
- $\mathcal{C}_n^4 = \mathcal{C}_n^5 = \mathcal{C}_n^6 = \cdots$

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	?	?	?	?
$\mathbb{R}$	?	?	?	?
k	?	?	?	?

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	?	?	?
$\mathbb{R}$	✓	?	?	?
k	✓	?	?	?

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	?	?	✓
$\mathbb{R}$	✓	?	?	✓
k	✓	?	?	?

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	**?**	?	✓
$\mathbb{R}$	✓	**?**	?	✓
k	✓	**?**	?	?

The distance 2 commuting set when $n = 3$

Let $A, B, C \in \mathrm{Mat}_3(k)$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

The distance 2 commuting set when $n = 3$

Let $A, B, C \in \mathrm{Mat}_3(k)$ with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \; C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Then $A \leftrightarrow C$ if and only if the following all hold:

$$\begin{aligned}
a_{13}c_{31} + a_{12}c_{21} + a_{11}c_{11} - c_{13}a_{31} - c_{12}a_{21} - c_{11}a_{11} &= 0 \\
a_{13}c_{32} + a_{12}c_{22} + a_{11}c_{12} - c_{13}a_{32} - c_{12}a_{22} - c_{11}a_{12} &= 0 \\
a_{13}c_{33} + a_{12}c_{23} + a_{11}c_{13} - c_{13}a_{33} - c_{12}a_{23} - c_{11}a_{13} &= 0 \\
a_{23}c_{31} + a_{22}c_{21} + a_{21}c_{11} - c_{23}a_{31} - c_{22}a_{21} - c_{21}a_{11} &= 0 \\
a_{23}c_{32} + a_{22}c_{22} + a_{21}c_{12} - c_{23}a_{32} - c_{22}a_{22} - c_{21}a_{12} &= 0 \\
a_{23}c_{33} + a_{22}c_{23} + a_{21}c_{13} - c_{23}a_{33} - c_{22}a_{23} - c_{21}a_{13} &= 0 \\
a_{33}c_{31} + a_{32}c_{21} + a_{31}c_{11} - c_{33}a_{31} - c_{32}a_{21} - c_{31}a_{11} &= 0 \\
a_{33}c_{32} + a_{32}c_{22} + a_{31}c_{12} - c_{33}a_{32} - c_{32}a_{22} - c_{31}a_{12} &= 0 \\
a_{33}c_{33} + a_{32}c_{23} + a_{31}c_{13} - c_{33}a_{33} - c_{32}a_{23} - c_{31}a_{13} &= 0
\end{aligned}$$

The distance 2 commuting set when $n = 3$

So $A \leftrightarrow C$ if and only if

$$\begin{bmatrix} 0 & -a_{21} & 0 & -a_{13} & a_{12} & 0 & a_{13} & 0 & 0 \\ -a_{12} & a_{11}-a_{22} & -a_{32} & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\ -a_{13} & -a_{23} & a_{11}-a_{33} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ a_{21} & 0 & 0 & a_{22}-a_{11} & -a_{21} & -a_{31} & a_{23} & 0 & 0 \\ 0 & a_{21} & 0 & -a_{12} & 0 & -a_{32} & 0 & a_{23} & 0 \\ 0 & 0 & a_{21} & -a_{13} & -a_{23} & a_{22}-a_{33} & 0 & 0 & a_{23} \\ a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33}-a_{11} & -a_{21} & -a_{31} \\ 0 & a_{31} & 0 & 0 & a_{32} & 0 & -a_{12} & a_{33}-a_{22} & -a_{32} \\ 0 & 0 & a_{31} & 0 & 0 & a_{32} & -a_{13} & -a_{23} & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The distance 2 commuting set when $n = 3$

So $A \leftrightarrow C$ if and only if

$$\begin{bmatrix} 0 & -a_{21} & 0 & -a_{13} & a_{12} & 0 & a_{13} & 0 & 0 \\ -a_{12} & a_{11}-a_{22} & -a_{32} & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\ -a_{13} & -a_{23} & a_{11}-a_{33} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\ a_{21} & 0 & 0 & a_{22}-a_{11} & -a_{21} & -a_{31} & a_{23} & 0 & 0 \\ 0 & a_{21} & 0 & -a_{12} & 0 & -a_{32} & 0 & a_{23} & 0 \\ 0 & 0 & a_{21} & -a_{13} & -a_{23} & a_{22}-a_{33} & 0 & 0 & a_{23} \\ a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33}-a_{11} & -a_{21} & -a_{31} \\ 0 & a_{31} & 0 & 0 & a_{32} & 0 & -a_{12} & a_{33}-a_{22} & -a_{32} \\ 0 & 0 & a_{31} & 0 & 0 & a_{32} & -a_{13} & -a_{23} & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similarly, $B \leftrightarrow C$ if and only if

$$\begin{bmatrix} 0 & -b_{21} & 0 & -b_{13} & b_{12} & 0 & b_{13} & 0 & 0 \\ -b_{12} & b_{11}-b_{22} & -b_{32} & 0 & b_{12} & 0 & 0 & b_{13} & 0 \\ -b_{13} & -b_{23} & b_{11}-b_{33} & 0 & 0 & b_{12} & 0 & 0 & b_{13} \\ b_{21} & 0 & 0 & b_{22}-b_{11} & -b_{21} & -b_{31} & b_{23} & 0 & 0 \\ 0 & b_{21} & 0 & -b_{12} & 0 & -b_{32} & 0 & b_{23} & 0 \\ 0 & 0 & b_{21} & -b_{13} & -b_{23} & b_{22}-b_{33} & 0 & 0 & b_{23} \\ b_{31} & 0 & 0 & b_{32} & 0 & 0 & b_{33}-b_{11} & -b_{21} & -b_{31} \\ 0 & b_{31} & 0 & 0 & b_{32} & 0 & -b_{12} & b_{33}-b_{22} & -b_{32} \\ 0 & 0 & b_{31} & 0 & 0 & b_{32} & -b_{13} & -b_{23} & 0 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The distance 2 commuting set when $n = 3$

So $A \leftrightarrow C \leftrightarrow B$ if and only if

$$
\begin{bmatrix}
0 & -a_{21} & 0 & -a_{13} & a_{12} & 0 & a_{13} & 0 & 0 \\
-a_{12} & a_{11}-a_{22} & -a_{32} & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\
-a_{13} & -a_{23} & a_{11}-a_{33} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\
a_{21} & 0 & 0 & a_{22}-a_{11} & -a_{21} & -a_{31} & a_{23} & 0 & 0 \\
0 & a_{21} & 0 & -a_{12} & 0 & -a_{32} & 0 & a_{23} & 0 \\
0 & 0 & a_{21} & -a_{13} & -a_{23} & a_{22}-a_{33} & 0 & 0 & a_{23} \\
a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33}-a_{11} & -a_{21} & -a_{31} \\
0 & a_{31} & 0 & 0 & a_{32} & 0 & -a_{12} & a_{33}-a_{22} & -a_{32} \\
0 & 0 & a_{31} & 0 & 0 & a_{32} & -a_{13} & -a_{23} & 0 \\
0 & -b_{21} & 0 & -b_{13} & b_{12} & 0 & b_{13} & 0 & 0 \\
-b_{12} & b_{11}-b_{22} & -b_{32} & 0 & b_{12} & 0 & 0 & b_{13} & 0 \\
-b_{13} & -b_{23} & b_{11}-b_{33} & 0 & 0 & b_{12} & 0 & 0 & b_{13} \\
b_{21} & 0 & 0 & b_{22}-b_{11} & -b_{21} & -b_{31} & b_{23} & 0 & 0 \\
0 & b_{21} & 0 & -b_{12} & 0 & -b_{32} & 0 & b_{23} & 0 \\
0 & 0 & b_{21} & -b_{13} & -b_{23} & b_{22}-b_{33} & 0 & 0 & b_{23} \\
b_{31} & 0 & 0 & b_{32} & 0 & 0 & b_{33}-b_{11} & -b_{21} & -b_{31} \\
0 & b_{31} & 0 & 0 & b_{32} & 0 & -b_{12} & b_{33}-b_{22} & -b_{32} \\
0 & 0 & b_{31} & 0 & 0 & b_{32} & -b_{13} & -b_{23} & 0
\end{bmatrix}
\begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The distance 2 commuting set when $n = 3$

So $A \leftrightarrow C \leftrightarrow B$ if and only if

$$\begin{bmatrix}
0 & -a_{21} & 0 & -a_{13} & a_{12} & 0 & a_{13} & 0 & 0 \\
-a_{12} & a_{11}-a_{22} & -a_{32} & 0 & a_{12} & 0 & 0 & a_{13} & 0 \\
-a_{13} & -a_{23} & a_{11}-a_{33} & 0 & 0 & a_{12} & 0 & 0 & a_{13} \\
a_{21} & 0 & 0 & a_{22}-a_{11} & -a_{21} & -a_{31} & a_{23} & 0 & 0 \\
0 & a_{21} & 0 & -a_{12} & 0 & -a_{32} & 0 & a_{23} & 0 \\
0 & 0 & a_{21} & -a_{13} & -a_{23} & a_{22}-a_{33} & 0 & 0 & a_{23} \\
a_{31} & 0 & 0 & a_{32} & 0 & 0 & a_{33}-a_{11} & -a_{21} & -a_{31} \\
0 & a_{31} & 0 & 0 & a_{32} & 0 & -a_{12} & a_{33}-a_{22} & -a_{32} \\
0 & 0 & a_{31} & 0 & 0 & a_{32} & -a_{13} & -a_{23} & 0 \\
0 & -b_{21} & 0 & -b_{13} & b_{12} & 0 & b_{13} & 0 & 0 \\
-b_{12} & b_{11}-b_{22} & -b_{32} & 0 & b_{12} & 0 & 0 & b_{13} & 0 \\
-b_{13} & -b_{23} & b_{11}-b_{33} & 0 & 0 & b_{12} & 0 & 0 & b_{13} \\
b_{21} & 0 & 0 & b_{22}-b_{11} & -b_{21} & -b_{31} & b_{23} & 0 & 0 \\
0 & b_{21} & 0 & -b_{12} & 0 & -b_{32} & 0 & b_{23} & 0 \\
0 & 0 & b_{21} & -b_{13} & -b_{23} & b_{22}-b_{33} & 0 & 0 & b_{23} \\
b_{31} & 0 & 0 & b_{32} & 0 & 0 & b_{33}-b_{11} & -b_{21} & -b_{31} \\
0 & b_{31} & 0 & 0 & b_{32} & 0 & -b_{12} & b_{33}-b_{22} & -b_{32} \\
0 & 0 & b_{31} & 0 & 0 & b_{32} & -b_{13} & -b_{23} & 0
\end{bmatrix}
\begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{31} \\ c_{32} \\ c_{33} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Write this equation as $M_{A,B}\mathbf{c} = \mathbf{0}$.

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$,

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$, **such that** C **is a nonscalar matrix**.

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$, **such that** C **is a nonscalar matrix**.

The equation

$$M_{A,B}\mathbf{z} = \mathbf{0}$$

already has a one-dimensional family of solutions: the scalar matrices. We need to have more solutions than that.

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$, **such that C is a nonscalar matrix**.

The equation

$$M_{A,B}\mathbf{z} = \mathbf{0}$$

already has a one-dimensional family of solutions: the scalar matrices. We need to have more solutions than that.

$$d(A, B) \leq 2 \Leftrightarrow \mathrm{null}(M_{A,B}) \geq 2$$

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$, **such that** C **is a nonscalar matrix**.

The equation

$$M_{A,B}\mathbf{z} = \mathbf{0}$$

already has a one-dimensional family of solutions: the scalar matrices. We need to have more solutions than that.

$$\begin{aligned} d(A, B) \leq 2 &\Leftrightarrow \mathrm{null}(M_{A,B}) \geq 2 \\ &\Leftrightarrow \mathrm{rank}(M_{A,B}) \leq 7 \end{aligned}$$

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$, **such that C is a nonscalar matrix**.

The equation

$$M_{A,B}\mathbf{z} = \mathbf{0}$$

already has a one-dimensional family of solutions: the scalar matrices. We need to have more solutions than that.

$$\begin{aligned} d(A, B) \leq 2 &\Leftrightarrow \mathrm{null}(M_{A,B}) \geq 2 \\ &\Leftrightarrow \mathrm{rank}(M_{A,B}) \leq 7 \\ &\Leftrightarrow \text{all } 8 \times 8 \text{ minors of } M_{A,B} \text{ are } 0. \end{aligned}$$

The distance 2 commuting set when $n = 3$

We have $A \leftrightarrow C \leftrightarrow B$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$.

So $d(A, B) \leq 2$ if and only if $M_{A,B}\mathbf{c} = \mathbf{0}$ for some $\mathbf{c}$, **such that C is a nonscalar matrix**.

The equation

$$M_{A,B}\mathbf{z} = \mathbf{0}$$

already has a one-dimensional family of solutions: the scalar matrices. We need to have more solutions than that.

$$\begin{aligned} d(A, B) \leq 2 &\Leftrightarrow \text{null}(M_{A,B}) \geq 2 \\ &\Leftrightarrow \text{rank}(M_{A,B}) \leq 7 \\ &\Leftrightarrow \text{all } 8 \times 8 \text{ minors of } M_{A,B} \text{ are } 0. \end{aligned}$$

This last condition is a bunch of polynomials in the entries of A and B!!

The distance 2 commuting set when $n = 3$

So the set $\mathcal{C}_3^2 = \{(A, B) \mid d(A, B) \leq 2\} \subset \mathrm{Mat}_3(k)^2$ is a variety, defined by the vanishing of all 8×8 minors of the matrix

$$\begin{pmatrix}
0 & -x_{21} & 0 & -x_{13} & x_{12} & 0 & x_{13} & 0 & 0 \\
-x_{12} & x_{11}-x_{22} & -x_{32} & 0 & x_{12} & 0 & 0 & x_{13} & 0 \\
-x_{13} & -x_{23} & x_{11}-x_{33} & 0 & 0 & x_{12} & 0 & 0 & x_{13} \\
x_{21} & 0 & 0 & x_{22}-x_{11} & -x_{21} & -x_{31} & x_{23} & 0 & 0 \\
0 & x_{21} & 0 & -x_{12} & 0 & -x_{32} & 0 & x_{23} & 0 \\
0 & 0 & x_{21} & -x_{13} & -x_{23} & x_{22}-x_{33} & 0 & 0 & x_{23} \\
x_{31} & 0 & 0 & x_{32} & 0 & 0 & x_{33}-x_{11} & -x_{21} & -x_{31} \\
0 & x_{31} & 0 & 0 & x_{32} & 0 & -x_{12} & x_{33}-x_{22} & -x_{32} \\
0 & 0 & x_{31} & 0 & 0 & x_{32} & -x_{13} & -x_{23} & 0 \\
0 & -y_{21} & 0 & -y_{13} & y_{12} & 0 & y_{13} & 0 & 0 \\
-y_{12} & y_{11}-y_{22} & -y_{32} & 0 & y_{12} & 0 & 0 & y_{13} & 0 \\
-y_{13} & -y_{23} & y_{11}-y_{33} & 0 & 0 & y_{12} & 0 & 0 & y_{13} \\
y_{21} & 0 & 0 & y_{22}-y_{11} & -y_{21} & -y_{31} & y_{23} & 0 & 0 \\
0 & y_{21} & 0 & -y_{12} & 0 & -y_{32} & 0 & y_{23} & 0 \\
0 & 0 & y_{21} & -y_{13} & -y_{23} & y_{22}-y_{33} & 0 & 0 & y_{23} \\
y_{31} & 0 & 0 & y_{32} & 0 & 0 & y_{33}-y_{11} & -y_{21} & -y_{31} \\
0 & y_{31} & 0 & 0 & y_{32} & 0 & -y_{12} & y_{33}-y_{22} & -y_{32} \\
0 & 0 & y_{31} & 0 & 0 & y_{32} & -y_{13} & -y_{23} & 0
\end{pmatrix}$$

The distance 2 commuting set for $n \geq 3$

Theorem (Elyze-Guterman-M., 2018)

Let k be **any** *field. Then $\mathcal{C}_n^2$ is a variety.*

The distance 2 commuting set for $n \geq 3$

Theorem (Elyze-Guterman-M., 2018)

Let k be **any** *field. Then $\mathcal{C}_n^2$ is a variety.*

Let $A, B \in \mathrm{Mat}_n(k)$. Build $M_{A,B}$ as before.

The distance 2 commuting set for $n \geq 3$

Theorem (Elyze-Guterman-M., 2018)

Let k be **any** *field. Then $\mathcal{C}_n^2$ is a variety.*

Let $A, B \in \mathrm{Mat}_n(k)$. Build $M_{A,B}$ as before. Then

$$\begin{aligned} d(A,B) \leq 2 &\Leftrightarrow \mathrm{null}(M_{A,B}) \geq 2 \\ &\Leftrightarrow \mathrm{rank}(M_{A,B}) \leq (n^2 - 2) \\ &\Leftrightarrow \text{all } (n^2-1) \times (n^2-1) \text{ minors of } M_{A,B} \text{ are } 0 \end{aligned}$$

The distance 2 commuting set for $n \geq 3$

Theorem (Elyze-Guterman-M., 2018)

Let k be **any** *field. Then $\mathcal{C}_n^2$ is a variety.*

Let $A, B \in \mathrm{Mat}_n(k)$. Build $M_{A,B}$ as before. Then

$$\begin{aligned} d(A,B) \leq 2 &\Leftrightarrow \mathrm{null}(M_{A,B}) \geq 2 \\ &\Leftrightarrow \mathrm{rank}(M_{A,B}) \leq (n^2 - 2) \\ &\Leftrightarrow \text{all } (n^2-1) \times (n^2-1) \text{ minors of } M_{A,B} \text{ are } 0 \end{aligned}$$

The final condition can be rephrased as a set of polynomials equalling 0 when the coordinates of A, B are plugged in.

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	?	?	✓
$\mathbb{R}$	✓	?	?	✓
k	✓	?	?	?

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	✓	?	✓
$\mathbb{R}$	✓	✓	?	✓
k	✓	✓	?	?

When is $\mathcal{C}_n^d$ a variety?

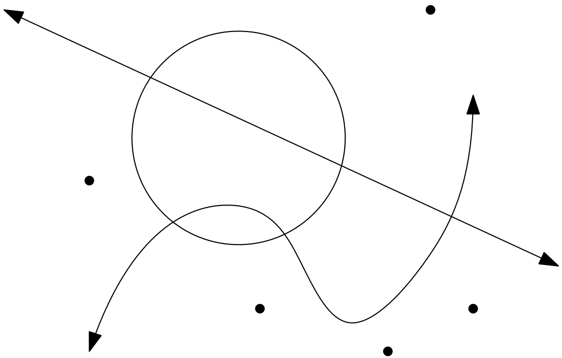
Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	✓	✓	✓
$\mathbb{R}$	✓	✓	?	✓
k	✓	✓	?	?

When is $\mathcal{C}_n^d$ a variety?

Field	$d \leq 1$	$d = 2$	$d = 3$	$d \geq 4$
$\mathbb{C}$	✓	✓	✓	✓
$\mathbb{R}$	✓	✓	✓	✓
k	✓	✓	?	?

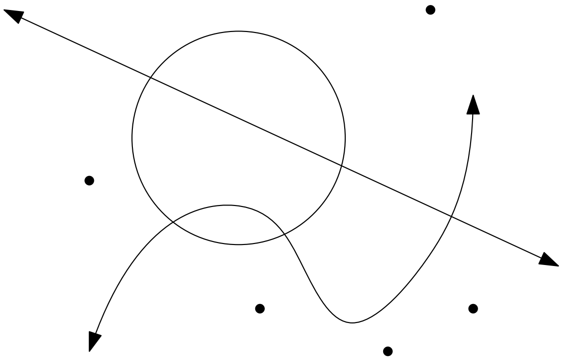
Most pairs of matrices are far apart on $\Gamma(n, k)$

When working over $\mathbb{R}$ or $\mathbb{C}$, all varieties (besides the entire space) have measure 0.



Most pairs of matrices are far apart on $\Gamma(n, k)$

When working over $\mathbb{R}$ or $\mathbb{C}$, all varieties (besides the entire space) have measure 0.



Corollary (Elyze-Guterman-M., 2018)

If A and B are "random" matrices over $\mathbb{R}$ or $\mathbb{C}$, then $d(A, B) = 4$.

Some open questions

Is $\mathcal{C}_n^d$ every **not** a variety?

Some open questions

Is $\mathcal{C}_n^d$ every **not** a variety?

- Some nice fields to consider: $\mathbb{Q}$, or a field with $4 < \mathrm{diam}(\Gamma(n, k)) \leq 6$.

Some open questions

Is $\mathcal{C}_n^d$ every **not** a variety?

- Some nice fields to consider: $\mathbb{Q}$, or a field with $4 < \mathrm{diam}(\Gamma(n,k)) \leq 6$.

Over $\mathbb{C}$, what are the **properties** of $\mathcal{C}_n^d$?

Some open questions

Is $\mathcal{C}_n^d$ every **not** a variety?

- Some nice fields to consider: $\mathbb{Q}$, or a field with $4 < \mathrm{diam}(\Gamma(n, k)) \leq 6$.

Over $\mathbb{C}$, what are the **properties** of $\mathcal{C}_n^d$?

- Dimension? Degree?

Some open questions

Is $\mathcal{C}_n^d$ every **not** a variety?

- Some nice fields to consider: $\mathbb{Q}$, or a field with $4 < \mathrm{diam}(\Gamma(n, k)) \leq 6$.

Over $\mathbb{C}$, what are the **properties** of $\mathcal{C}_n^d$?

- Dimension? Degree?
- Partial results: $\dim(\mathcal{C}_n^2) \geq 2n^2 - 2n + 2$, $\dim(\mathcal{C}_n^3) \geq 2n^2 - 3$.

Some open questions

Is $\mathcal{C}_n^d$ every **not** a variety?

- Some nice fields to consider: $\mathbb{Q}$, or a field with $4 < \mathrm{diam}(\Gamma(n,k)) \leq 6$.

Over $\mathbb{C}$, what are the **properties** of $\mathcal{C}_n^d$?

- Dimension? Degree?
- Partial results: $\dim(\mathcal{C}_n^2) \geq 2n^2 - 2n + 2$, $\dim(\mathcal{C}_n^3) \geq 2n^2 - 3$.

What are nice defining sets of polynomials?

Some open questions

$$
\begin{aligned}
&x_{3,1}x_{3,2}y_{1,1}y_{1,3}y_{2,2} - x_{3,1}^2 y_{1,2}y_{1,3}y_{2,2} + x_{3,2}^2 y_{1,3}y_{2,1}y_{2,2} - x_{3,1}x_{3,2}y_{1,3}y_{2,2}^2 - x_{3,1}x_{3,2}y_{1,1}y_{1,2}y_{2,3} \\
&+ x_{3,1}^2 y_{1,2}^2 y_{2,3} - x_{3,2}^2 y_{1,2}y_{2,1}y_{2,3} + x_{3,1}x_{3,2}y_{1,2}y_{2,2}y_{2,3} - x_{2,3}x_{3,1}y_{1,2}^2 y_{3,1} + x_{2,2}x_{3,1}y_{1,2}y_{1,3}y_{3,1} \\
&- x_{3,1}x_{3,3}y_{1,2}y_{1,3}y_{3,1} + x_{3,1}x_{3,2}y_{1,3}^2 y_{3,1} + x_{1,3}x_{3,2}y_{1,2}y_{2,1}y_{3,1} - x_{1,2}x_{3,2}y_{1,3}y_{2,1}y_{3,1} - x_{1,3}x_{3,2}y_{1,1}y_{2,2}y_{3,1} \\
&+ x_{1,3}x_{3,1}y_{1,2}y_{2,2}y_{3,1} - x_{2,3}x_{3,2}y_{1,2}y_{2,2}y_{3,1} + x_{1,3}x_{3,2}y_{2,2}^2 y_{3,1} + x_{1,2}x_{3,2}y_{1,1}y_{2,3}y_{3,1} - x_{1,2}x_{3,1}y_{1,2}y_{2,3}y_{3,1} \\
&+ x_{2,2}x_{3,2}y_{1,2}y_{2,3}y_{3,1} - x_{3,2}x_{3,3}y_{1,2}y_{2,3}y_{3,1} + x_{3,2}^2 y_{1,3}y_{2,3}y_{3,1} - x_{1,2}x_{3,2}y_{2,2}y_{2,3}y_{3,1} - x_{1,3}x_{2,2}y_{1,2}y_{3,1}^2 \\
&+ x_{1,2}x_{2,3}y_{1,2}y_{3,1}^2 + x_{1,3}x_{3,3}y_{1,2}y_{3,1}^2 - x_{1,3}x_{3,2}y_{1,3}y_{3,1}^2 + x_{2,3}x_{3,1}y_{1,1}y_{1,2}y_{3,2} - x_{2,2}x_{3,1}y_{1,1}y_{1,3}y_{3,2} \\
&+ x_{3,1}x_{3,3}y_{1,1}y_{1,3}y_{3,2} - x_{3,1}^2 y_{1,3}^2 y_{3,2} - x_{1,3}x_{3,1}y_{1,2}y_{2,1}y_{3,2} + x_{2,3}x_{3,2}y_{1,2}y_{2,1}y_{3,2} + x_{1,2}x_{3,1}y_{1,3}y_{2,1}y_{3,2} \\
&- x_{2,2}x_{3,2}y_{1,3}y_{2,1}y_{3,2} + x_{3,2}x_{3,3}y_{1,3}y_{2,1}y_{3,2} + x_{2,2}x_{3,1}y_{1,3}y_{2,2}y_{3,2} - x_{3,1}x_{3,3}y_{1,3}y_{2,2}y_{3,2} \\
&- x_{1,3}x_{3,2}y_{2,1}y_{2,2}y_{3,2} - x_{2,2}x_{3,1}y_{1,2}y_{2,3}y_{3,2} + x_{3,1}x_{3,3}y_{1,2}y_{2,3}y_{3,2} - x_{3,1}x_{3,2}y_{1,3}y_{2,3}y_{3,2} \\
&+ x_{1,2}x_{3,2}y_{2,1}y_{2,3}y_{3,2} + x_{1,3}x_{2,2}y_{1,1}y_{3,1}y_{3,2} - x_{1,2}x_{2,3}y_{1,1}y_{3,1}y_{3,2} - x_{1,3}x_{3,3}y_{1,1}y_{3,1}y_{3,2} \\
&+ x_{1,3}x_{3,1}y_{1,3}y_{3,1}y_{3,2} - x_{2,3}x_{3,2}y_{1,3}y_{3,1}y_{3,2} - x_{1,3}x_{2,2}y_{2,2}y_{3,1}y_{3,2} + x_{1,2}x_{2,3}y_{2,2}y_{3,1}y_{3,2} \\
&+ x_{1,3}x_{3,3}y_{2,2}y_{3,1}y_{3,2} + x_{2,3}x_{3,1}y_{1,3}y_{3,2}^2 + x_{1,3}x_{2,2}y_{2,1}y_{3,2}^2 - x_{1,2}x_{2,3}y_{2,1}y_{3,2}^2 - x_{1,3}x_{3,3}y_{2,1}y_{3,2}^2 \\
&- x_{3,1}x_{3,2}y_{1,1}y_{1,3}y_{3,3} + x_{3,1}^2 y_{1,2}y_{1,3}y_{3,3} - x_{3,2}^2 y_{1,3}y_{2,1}y_{3,3} + x_{3,1}x_{3,2}y_{1,3}y_{2,2}y_{3,3} + x_{1,3}x_{3,2}y_{1,1}y_{3,1}y_{3,3} \\
&- x_{1,3}x_{3,1}y_{1,2}y_{3,1}y_{3,3} + x_{2,3}x_{3,2}y_{1,2}y_{3,1}y_{3,3} - x_{1,3}x_{3,2}y_{2,2}y_{3,1}y_{3,3} - x_{2,3}x_{3,1}y_{1,2}y_{3,2}y_{3,3} \\
&+ x_{1,3}x_{3,2}y_{2,1}y_{3,2}y_{3,3}
\end{aligned}
$$

Thank you!

- Akbari, S.; Bidkhori, H.; Mohammadian, A. *Commuting graphs of matrix algebras*. Comm. Algebra **36** (2008), no. 11, 4020–4031.
- Akbari, S.; Mohammadian, A.; Radjavi, H.; Raja, P. *On the diameters of commuting graphs*. Linear Algebra Appl. **418** (2006), no. 1, 161–176.
- Akbari, S.; Raja, P. *Commuting graphs of some subsets in simple rings*. Linear Algebra Appl. **416** (2006), no. 2-3, 1038–1047.
- Elyze, M.; Guterman, A.; Morrison, R. *Higher distance commuting varieties*. arxiv:1811.09553
- Shitov, Y.. *A matrix ring with commuting graph of maximal diameter*. J. Combin. Theory Ser. A **141** (2016), 127–135.
- Shitov, Y. *Distances on the commuting graph of the ring of real matrices*. (Russian) ; translated from Mat. Zametki **103** (2018), no. 5, 765–768 Math. Notes **103** (2018), no. 5-6, 832–835.

Proof in the algebraically closed case

We have $d(A, B) \leq 2$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\}$$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

is Zariski closed.

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \,|\, A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

is Zariski closed. Let π project onto $\mathrm{Mat}_n(k)^2$, so π is proper.

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

is Zariski closed. Let π project onto $\mathrm{Mat}_n(k)^2$, so π is proper.

Then $\pi(\mathcal{T})$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

is Zariski closed. Let π project onto $\mathrm{Mat}_n(k)^2$, so π is proper.

Then $\pi(\mathcal{T}) = \mathcal{C}_n^2$

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

is Zariski closed. Let π project onto $\mathrm{Mat}_n(k)^2$, so π is proper.

Then $\pi(\mathcal{T}) = \mathcal{C}_n^2$ is Zariski closed.

Proof in the algebraically closed case

We have $d(A, B) \leq 2$
$\Leftrightarrow$ there exists a nonscalar matrix C with $A \leftrightarrow C \leftrightarrow B$
$\Leftrightarrow$ there exists a nonscalar D with $\mathrm{tr}(D) = 0$ and $A \leftrightarrow D \leftrightarrow B$

The set of all such D (up to scaling) can be identified with a projective space $\mathbb{P}^{n^2-2}$.

The set

$$\mathcal{T} := \{(A, B, D) \mid A \leftrightarrow D \leftrightarrow B\} \subset \mathrm{Mat}_n(k)^2 \times \mathbb{P}^{n^2-2}$$

is Zariski closed. Let π project onto $\mathrm{Mat}_n(k)^2$, so π is proper.

Then $\pi(\mathcal{T}) = \mathcal{C}_n^2$ is Zariski closed.

The same argument works for $\mathcal{C}_n^3$.

$\mathcal{C}_n^3$ for real closed fields

Let k be real closed. We know $\mathcal{C}_n^3\left(\overline{k}\right)$ is a variety.

$\mathcal{C}_n^3$ for real closed fields

Let k be real closed. We know $\mathcal{C}_n^3\left(\overline{k}\right)$ is a variety.

Key ideas to extend to $\mathcal{C}_n^3(k)$:

$\mathcal{C}_n^3$ for real closed fields

Let k be real closed. We know $\mathcal{C}_n^3\left(\overline{k}\right)$ is a variety.

Key ideas to extend to $\mathcal{C}_n^3(k)$:

- If $d_{\overline{k}}(A, B) \leq 3$, then $d_k(A, B) \leq 3$.

$\mathcal{C}_n^3$ for real closed fields

Let k be real closed. We know $\mathcal{C}_n^3\left(\overline{k}\right)$ is a variety.

Key ideas to extend to $\mathcal{C}_n^3(k)$:

- If $d_{\overline{k}}(A, B) \leq 3$, then $d_k(A, B) \leq 3$. So,

$$\mathcal{C}_n^3(k) = \mathcal{C}_n^3\left(\overline{k}\right) \cap \mathrm{Mat}_n(k)^2$$

$\mathcal{C}_n^3$ for real closed fields

Let k be real closed. We know $\mathcal{C}_n^3\left(\overline{k}\right)$ is a variety.

Key ideas to extend to $\mathcal{C}_n^3(k)$:

- If $d_{\overline{k}}(A, B) \leq 3$, then $d_k(A, B) \leq 3$. So,

$$\mathcal{C}_n^3(k) = \mathcal{C}_n^3\left(\overline{k}\right) \cap \mathrm{Mat}_n(k)^2$$

- If k is a real closed field and $V \subset \overline{k}^m$ is an affine variety, then so is $V \cap k^m$.