History Dependent Stochastic Processes and Applications to Finance

by
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Abstract

In this paper we focus on properties of discretized random walk, the stochastic processes achieved in their limit, and applications of these processes to finance. We go through a brief foray into probability spaces and σ-fields, discrete and continuous random walks, stochastic process and Itô calculus, and Brownian motion. We study the properties that make Brownian motion unique and how it is constructed as a limit of a discrete independent random walk. Using this understanding we propose a different kind of random walk that remembers the past. We investigate this new random walk and find properties similar to those of the symmetric random walk.

We look at a well-studied stochastic process called fractional Brownian motion, which uses the Hurst parameter to remember past performance. Using Brownian motion and fractional Brownian motion to model stock behavior, we then detail the famous Black-Scholes options formula and a fractional Black-Scholes model. We compare their performance and accuracy through the observation of twenty different stocks in the market. Finally we discuss under what circumstances is the fractional model more accurate at predicting stock price compared to the standard model and explanations for why this might occur.

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1 Introduction

The end goal of this paper is to examine history dependent random walks and to see how we can apply them to finance using the Black-Scholes model. To achieve this goal we studied in-depth probability spaces, random variables, and properties of random walk. We start with a simple symmetric random walk.

Definition 1.0.1. A simple random walk is a sequence of independent Bernoulli random variables $X_1, X_2, ...$ Its sum is given by

$$S_n = \sum_{i=1}^{n} X_i.$$

We can manipulate this random walk by now taking step size $1/\sqrt{k}$ and take steps at time $1/k$. Therefore at time t the random walk is at step $n = tk$. Doing this we show that as $k \to \infty$, this random walk converges to the stochastic process of Brownian motion in distribution.

Now that we know how Brownian motion is constructed as a limit of random walks, we proposed our own random walk and attempted to take its limit. We define this new random walk as

Definition 1.0.2. Our random walk X_n^1 is defined by

$$X_0^1 = \{1 \text{ or } -1\} \text{ with probabilities } \left\{\frac{1}{2} \text{ and } \frac{1}{2}\right\}$$

and

$$X_n^1 = \{1 \text{ or } \ -1\} \text{ with probabilities} \left\{\frac{1}{2} + pX_{n-1}^1 \text{ and } \frac{1}{2} - pX_{n-1}^1\right\}$$

where $-\frac{1}{2} < p < \frac{1}{2}$.

We take the limit here in the same way that we did with the simple random walk to achieve Brownian motion, and get a stochastic process with similar properties. This new stochastic process has a variance depending on p, stationary increments, and has a normal Gaussian distribution. The difference between this stochastic process and Brownian motion is the variance and the lack of independent increments since this new process relies on the past.

Next we examine how we can apply these processes to finance, specifically the stock market. Brownian motion is used to model the randomness in stock prices, and helps to give us the Black-Scholes Options Pricing Formula. An option is a financial contract to purchase the given stock at an agreed upon price in the future no matter the price of the stock at that time. Such a contract comes at a price, and the Black-Scholes formula gives us a price of the option. However, as with Brownian motion, Black-Scholes prices the options as if the stocks follow a completely random process which might not be true in reality.

There is a well-studied process called fractional Brownian motion which remembers the past through a parameter H, called the Hurst parameter. The Hurst parameter is a statistical measure of a time series, here stock prices, ranging from 0 to 1. Values of $H > .5$ indicate some long-run dependence on the past and values of $H < .5$ indicate that the time series tends to revert to the mean, or do the opposite of what happens in the past. Using this process we can achieve a new options pricing formula, called the fractional Black-Scholes Options Pricing Formula. In this paper we compared, using 20 stocks of differing size, volatility, and Hurst parameter, the accuracy of each of the models in predicting stock prices and attempted to see which model was more accurate and under what conditions.

We used historic stock data for a one month period in January to February, 2015, and recorded the stock price every week and the predicted options price for that stock. Next we compare the predicted stock price to the actual stock price for the standard Black-Scholes equation and the fractional Black-Scholes equation. We found that for a Hurst parameter of less than .5 that the fractional Black-Scholes was a more accurate predictor of stock price and that for a Hurst parameter greater than .5 that the Black-Scholes was a more accurate predictor of stock price. We explore a couple of reasons for this in the paper. One such explanation is that for time frames of less than a year, if $H > .5$ then the fractional call price will always be decreasing. This means that it will predict an option price cheaper than that of the standard Black-Scholes. Economically the fractional Black-Scholes is pricing in risk and uncertainty with the Hurst parameter which is another intuitive reason behind the differences in the models.

2 Background

This section will provide some foundation for later work in the paper. We use Grimmett and Stirzaker's book on probability as the basis for the background ([3]).

2.1 Probability spaces

Here we examine general probability spaces and their properties which will be useful later for defining random variables which model the stock prices.

Definition 2.1.1. The *sample space* Ω of an experiment is the set of all possible outcomes.

Definition 2.1.2. A *σ-field* $\mathcal{F}$ on some probability space Ω is a collection of subsets of Ω satisfying the following condition:

1. It is not empty and $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
3. If $A_1, A_2, ... \in \mathcal{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \text{ and } \bigcap_{i=1}^{\infty} A_i \in \mathcal{F},$$

where the A_i are subsets of Ω.

Definition 2.1.3. A *probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P}: \mathcal{F} \to [0,1]$ satisfying

1. $\mathbb{P}(\emptyset) = 0, \quad \mathbb{P}(\Omega) = 1;$
2. If $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, in that $A_i \cap A_j = \emptyset$ for all pairs i, j satisfying $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 2.1.4. The triple $(\Omega, \mathcal{F}, \mathbb{P})$, comprising a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω, and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, is called a *probability space*.

Definition 2.1.5. Let A and B be different events that occur. A and B are called *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

More generally a family of events $\{A_i : i \in I\}$ is called independent if

$$\mathbb{P}\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} \mathbb{P}(A_i)$$

for all finite subsets J of I where I is the set of positive integers.

2.2 Random Variables

Random variables are integral to the discussion in this paper as they are used to approximate stock paths and in constructing Brownian motion and fractional Brownian motion.

Definition 2.2.1. A *random variable* is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$. This function is said to be *$\mathcal{F}$-measurable.*

Since a random variable is a function we need to define a way to describe properties of that function.

Definition 2.2.2. The *distribution function* of a random variable X is the function $F : \mathbb{R} \to [0, 1]$ given by $F(x) = \mathbb{P}(X \leq x)$.

Lemma 2.2.3. *A distribution function F has the following properties:*

1. $\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1.$
2. *If $x < y$ then $F(x) \leq F(y)$.*
3. *F is right-continuous, that is $F(x + h) \to F(x)$ as $h \to 0$.*

Most of the study of random variables centers around their distribution functions. We completely understand a random variable if we know its distribution function. There are two types of random variables studied in this paper, discrete and continuous.

2.2.1 Discrete Random Variables

Definition 2.2.4. The random variable X is called *discrete* if it takes values in some countable subset of $\mathbb{R}$.

Definition 2.2.5. The discrete random variable X has *probability or mass function* $f : \mathbb{R} \to [0, 1]$ given by $f(x) = \mathbb{P}(X = x)$.

The following lemma very simply describes some basic properties of the mass function.

Lemma 2.2.6. *The probability mass function $f : \mathbb{R} \to [0, 1]$ satisfies:*

1. *The set of x such that $f(x) \neq 0$ is countable,*
2. $\sum_i f(x_i) = 1$, *where $x_1, x_2, \ldots$ are the values of x such that $f(x) \neq 0$.*

Definition 2.2.7. The discrete random variables X and Y are *independent* if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all x and y.

This notion of independence is extremely useful in characterizing random walks which we will detail later in the paper. A quick, simple example of a discrete random variable is a Bernoulli variable.

Definition 2.2.8. A *Bernoulli variable* X takes the value of 1 with probability p and the value of 0 with probability $1 - p$.

The Bernoulli random variable is what we use to construct Brownian motion as a limit of random walks further on in the paper.

2.2.2 Continuous Random Variables

Definition 2.2.9. The random variable X is called *continuous* if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)du \quad x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the *probability or density function* of X.

Most of the theorems, lemmas, or definitions of discrete random variables either apply or have an analagous version for continuous random variables.

Lemma 2.2.10. *If X has density function f then:*

1. $\int_{-\infty}^{\infty} f(x)dx = 1$.
2. $\mathbb{P}(X = x) = 0$ *for all* $x \in \mathbb{R}$.
3. $\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$.

Definition 2.2.11. Continuous random variables X and Y are called *independent* if $\{X \leq x\}$ and $\{Y \leq y\}$ are independent events for all $x, y \in \mathbb{R}$.

Possibly the most famous example of a continuous random variable is the normal distribution which is defined as the following.

Definition 2.2.12. A random variable X with mean μ and variance σ^2 is normally distributed if it has density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

It is denoted by $N(\mu, \sigma^2)$. If $\mu = 0$ and $\sigma^2 = 1$, then we call it the standard normal distribution. The normal distribution is one of the properties used to define the stochastic process Brownian motion. A critical assumption of the Black-Scholes model is that stock prices are normally distributed, and therefore can be modeled using Brownian motion.

2.3 Expectation of Random Variables

Here we define and detail what different expected values of discrete and continuous random variables are.

2.3.1 Expectation of Discrete Random Variables

In constructing our new random walk we need to understand how to compute its expected value and variance, both of which rely on the expectation of the discrete random variables from which the walk is constructed.

Definition 2.3.1. The *mean value*, *expectation*, or *expected value* of the discrete random variable X with mass function f is defined to be

$$\mathbb{E}(X) = \sum_{x:f(x)>0} xf(x),$$

whenever this sum is absolutely convergent.

This sum has to be absolutely convergent in order to insure that $\mathbb{E}(X)$ is unchanged by reordering the x_i.

Lemma 2.3.2. *If the discrete random variable X has mass function f and $g : \mathbb{R} \to \mathbb{R}$, then*

$$\mathbb{E}(g(X)) = \sum_x g(x)f(x)$$

whenever this sum is absolutely convergent.

Definition 2.3.3. If k is a positive integer, the kth *moment* m_k of the discrete random variable X is defined to be $m_k = \mathbb{E}(X^k)$. The kth *central moment* μ_k is $\mu_k = \mathbb{E}((X - m_1)^k)$.

The most well-known and widely used moments are

$$m_1 = \mathbb{E}(X),$$

called the mean or expectation of X, and

$$\sigma_2 = \mathbb{E}((X - \mathbb{E}(X))^2),$$

called the variance of X. The variance of X can also be written as

$$\text{var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

2.3.2 Expectation of Continuous Random Variables

With discrete random variables, the expectation is an average of all the random variable's values weighted by its probability. With continuous random variables we need to use an integral.

Definition 2.3.4. The *expectation* of a continuous random variable X with density function f is given by
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx,$$
whenever such integral exists.

Theorem 2.3.5. *If X and $g(X)$ are continuous random variables, and X density function $f(x)$, then*
$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

The kth moment of a continuous random variable X is defined the same as with the discrete random variable.

2.3.3 Properties of the Expectation Operator

Now that we have defined what the expectation of a random variable is, it is useful to examine different properties and techniques in using the expectation operator $\mathbb{E}$ and calculating it. These properties are used to help compute the expected value and variance of Brownian motion, our new random walk, and fractional Brownian motion.

Theorem 2.3.6. *Let X, Y be random variables, then the expectation operator $\mathbb{E}$ has the following properties:*

1. *If $X \geq 0$ then $\mathbb{E}(X) \geq 0$.*
2. *If $a, b \in \mathbb{R}$ then $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$.*
3. *The random variable 1, taking the value 1 always, has expectation $\mathbb{E}(1) = 1$.*

Lemma 2.3.7. *If the random variables X and Y are independent then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.*

Definition 2.3.8. The random variables X and Y are *uncorrelated* if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

As the lemma above implies, independent random variables are uncorrelated. The converse, however, is not true.

Theorem 2.3.9. *For random variables X and Y,*

1. $\text{var}(aX) = a^2\text{var}(X)$ *for* $a \in \mathbb{R}$,

2. $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$ *if* X *and* Y *are uncorrelated.*

Proof. Using the linearity of $\mathbb{E}$,

$$\text{var}(aX) = \mathbb{E}\{(aX - \mathbb{E}(aX))^2\} = \mathbb{E}\{a^2(X - \mathbb{E}X)^2\}$$

$$= a^2\mathbb{E}\{(X - \mathbb{E}X)^2\} = a^2\text{var}(X).$$

For the second part, we have when X and Y are uncorrelated that

$$\begin{aligned}
\text{var}(X+Y) &= \mathbb{E}\{(X+Y-\mathbb{E}(X+Y))^2\} \\
&= \mathbb{E}[(X-\mathbb{E}X)^2 + 2(XY - \mathbb{E}(X)\mathbb{E}(Y)) + (Y-\mathbb{E}Y)^2] \\
&= \text{var}(X) + 2[\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)] + \text{var}(Y) \\
&= \text{var}(X) + \text{var}(Y).
\end{aligned}$$

□

2.4 Conditional Expectation of Random Variables

We'll start off with a general, concrete case of conditional expectation. Then we go into the conditional expectation of a random variable given a σ field which will become important later in understanding Itô integrals.

Definition 2.4.1. The conditional probability of event A given B is defined by:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Definition 2.4.2. Given that $\mathbb{P}(B) > 0$, we can define the conditional distribution function of a random variable X given B as

$$F_X(x|B) = \frac{\mathbb{P}(X \leq x, B)}{\mathbb{P}(B)}.$$

We can now begin to define the conditional expectation of a random variable.

Definition 2.4.3. The conditional expectation of X given B is

$$\mathbb{E}(X|B) = \frac{\mathbb{E}(XI_B)}{\mathbb{P}(B)}$$

where

$$I_B(\omega) = \begin{cases} 1 \ if \ \omega \in B \\ 0 \ if \ \omega \notin B. \end{cases}$$

If X is a discrete random variable, then the conditional expectation of X is

$$\mathbb{E}(X|B) = \sum_{k=1}^{\infty} x_k \mathbb{P}(X = x_k|B),$$

but if X has density function f_X, then the conditional expectation becomes

$$\frac{1}{\mathbb{P}(B)} \int_B x f_X(x) dx.$$

Now we can consider the conditional expectation of one variable on another. Consider the discrete random variable Y assuming distinct values on a set $\{y_i\} \subseteq \mathbb{R}$ and let

$$A_i = \{\omega \in \Omega : \ Y(\omega) = y_i\}.$$

Definition 2.4.4. For a random variable X on a probability space Ω with $\mathbb{E}|X| < \infty$ we define the conditional expectation of X given Y as the discrete random variable

$$\mathbb{E}(X|Y(\omega)) = \mathbb{E}(X|A_i) = \mathbb{E}(X|Y = y_i) \text{ for } \omega \in A_i.$$

2.5 Random Walk

A Random Walk is the process by which randomly moving objects move from where they started. It is a mathematical formalization of a path that consists of successive random steps. The stock prices we discuss in the applications section and in the Black-Scholes model are modeled as random walks. Knowing what a symmetric random walk is and its properties help in understanding and proving many analogous properties of our new random walk.

Definition 2.5.1. A simple random walk is a sequence of independent Bernoulli random variables $X_1, X_2, \ldots$. Its sum is given by

$$S_n = \sum_{i=1}^{n} X_i.$$

Lemma 2.5.2. *The simple random walk is spatially homogeneous; that is*

$$\mathbb{P}(S_n = j | S_0 = a) = \mathbb{P}(S_n = j + b | S_0 = a + b), \quad a, b, j \in \mathbb{Z}.$$

Proof. Both sides equal $\mathbb{P}(\sum_1^n X_i = j - a)$. $\square$

Lemma 2.5.3. *The simple random walk is temporally homogeneous; that is*

$$\mathbb{P}(S_n = j | S_0 = a) = \mathbb{P}(S_{m+n} = j | S_m = a).$$

Proof. The left- and right-hand sides satisfy

$$LHS = \mathbb{P}\left(\sum_{i=1}^{n} X_i = j - a\right) = \mathbb{P}\left(\sum_{i=m+1}^{m+n} X_i = j - a\right) = RHS.$$

$\square$

Lemma 2.5.4. *The simple random walk has the Markov property; that is*

$$\mathbb{P}(S_{m+n} = j | S_0, S_1, \ldots, S_m) = \mathbb{P}(S_{m+n} = j | S_m), \quad n \geq 0.$$

Proof. If the value of S_m is known, then the distribution of S_{m+n} depends only on the jumps $X_{m+1}, \ldots, X_{m+n}$ and cannot depend on any further information conceding the values of $S_0, S_1, \ldots, S_{m-1}$. $\square$

Definition 2.5.5. A random walk starts at $S_0 = a$ and after n steps ends at $S_n = b$. The probability that it follows a given path is $p^r q^l$ where r is the number of steps to the right and l is the number of steps to the left. Then

$$\mathbb{P}(S_n = b) = \binom{n}{\frac{1}{2}(n + b - a)} p^{\frac{1}{2}(n+b-a)} q^{\frac{1}{2}(n-b+a)}.$$

Next it is important to examine the number of possible paths that exist, and some properties of these paths. Let $N_n(a, b)$ be the number of possible paths from

$(0, a)$ to (n, b) and let $N_n^0(a, b)$ be the number of such paths which contain some point $(k, 0)$ on the x-axis.

Theorem 2.5.6. *If $a, b > 0$ then $N_n^0(a, b) = N_n(-a, b)$.*

Proof. Each path from $(0, -a)$ to (n, b) intersects the x-axis at some earliest point, call it $(k, 0)$. Reflect the segment of the path with $0 \leq x \leq k$ across the x-axis to obtain a path joining $(0, a)$ to (n, b) which intersects the x-axis. This gives a one-to-one correspondence of such paths. $\square$

Lemma 2.5.7. *We have $N_n(a, b) = \binom{n}{\frac{1}{2}(n+b-a)}$.*

Proof. Choose a path from $(0, a)$ to (n, b) and let α and β be the numbers of positive and negative steps, respectively, in this path. Then $\alpha + \beta = n$ and $\alpha - \beta = b - a$, so that $\alpha = \frac{1}{2}(n + b - a)$. The number of such paths is the number of ways of picking α positive steps from the n available. That is

$$N_n(a, b) = \binom{n}{\alpha} = \binom{n}{\frac{1}{2}(n + b - a)}.$$

$\square$

This next corollary, the Ballot Theorem, is a consequence of these two results.

Corollary 2.5.8. *If $b > 0$ then the number of paths from $(0, 0)$ to (n, b) which do not revisit the x-axis equals $(b/n)N_n(0, b)$.*

We also want to examine the chance that a random walk reaches a point b on the nth step, denoted $f_b(n)$.

Theorem 2.5.9. *The probability $f_b(n)$ that a random walk S hits the point b for the first time at the nth step, having started from 0, satisfies:*

$$f_b(n) = \frac{|b|}{n}\mathbb{P}(S_n = b) \ if \ n \geq 1.$$

Together with one of the previous theorems this gives us the probability of visiting some point b:

Theorem 2.5.10. *If $p = \frac{1}{2}$ and $S_0 = 0$ for any $b \neq 0$ the mean number μ_b of visits of the walk to the point b before returning to the origin equals 1.*

Proof. Let $f_b = \mathbb{P}(S_n = b \ for \ some \ n \geq 0)$. By conditioning on the value of S_1, we have that $f_b = \frac{1}{2}(f_{b+1} + f_{b-1})$ for $b>0$, with boundary condition $f_0 = 1$. The solution to this is $f_b = Ab + B$ for constants A, B. The unique solution in [0,1] with $f_0 = 1$ is given by $f_b = 1 \ \forall \ b \geq 0$. By symmetry $f_b = 1$ for $b \leq 0$. However $f_b = \mu_b$ for $b \neq 0$, and the claim follows. □

Finally we end the background on random walks with an important property of the symmetric random walk.

Theorem 2.5.11. *Suppose that $p = \frac{1}{2}$ and $S_0 = 0$. The probability that the last visit to 0 up to time $2n$ occurred at time $2k$ is:*

$$\mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2n-2k} = 0).$$

Proof.

$$\begin{aligned} \alpha_{2n} &= \mathbb{P}(S_{2k} = 0)\mathbb{P}(S_{2k+1}S_{2k+2} \cdots S_{2n} \neq 0 \mid S_{2k} = 0) \\ &= \mathbb{P}(S_{2k} = 0)\mathbb{P}(S_1 S_2 \cdots S_{2n-2k} \neq 0). \end{aligned}$$

Setting $m = n - k$, we have that

$$\begin{aligned} \mathbb{P}(S_1 S_2 \cdots S_{2m} \neq 0) &= 2\sum_{k=1}^{m} \frac{2k}{2m} \mathbb{P}(S_{2m} = 2k) \\ &= 2\sum_{k=1}^{m} \frac{2k}{2m} \binom{2m}{m+k} \left(\frac{1}{2}\right)^{2m} \\ &= 2\left(\frac{1}{2}\right)^{2m} \sum_{k=1}^{m} \left[\binom{2m-1}{m+k-1} - \binom{2m-1}{m+k}\right] \\ &= 2\left(\frac{1}{2}\right)^{2m} \binom{2m-1}{m} \\ &= \binom{2m}{m}\left(\frac{1}{2}\right)^{2m} = \mathbb{P}(S_{2m} = 0). \end{aligned}$$

□

3 Limit Theorems and Modes of Convergence

One of the important things we will need to study random variables and their values is how they converge, or what the limit of random variables are. This is critical in taking the limit of the random walk to get Brownian motion and fractional Brownian motion, and in attempting to take the limit of our new random walk.

3.1 Modes of Convergence

We outline here a few tools we can use to estimate random variables values. We enumerate the four most common modes of convergence.

Definition 3.1.1. Let $X_1, X_2, X_3, ...$ be random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1. $X_n \to X$ *almost surely*, written $X_n \xrightarrow{a.s.} X$, if $\{\omega \in \Omega : X_n(\omega) \to X(\omega)\ as\ n \to \infty\}$ is an event whose probability is 1.

2. $X_n \to X$ *in the rth mean*, where $r \geq 1$, written $X_n \xrightarrow{r} X$, if
$$\mathbb{E}(|X_n - X|^r) \to 0\ as\ n \to \infty.$$

3. $X_n \to X$ *in probability*, written $X_n \xrightarrow{P} X$, if
$$\mathbb{P}(|X_n - X| > \epsilon) \to 0\ as\ n \to \infty\ for\ all\ \epsilon > 0.$$

4. $X_n \to X$ *in distribution*, written $X_n \xrightarrow{D} X$, if
$$\mathbb{P}(X_n \leq x) \to \mathbb{P}(X \leq x)\ as\ n \to \infty.$$

Two important r's that we will use later are $r = 1$ and $r = 2$, we can write them as
$$X_n \xrightarrow{1} X, \text{ or } X_n \to X \text{ in mean}$$
$$X_n \xrightarrow{2} X, \text{ or } X_n \to X \text{ in mean-square.}$$

Given those modes of convergence we describe a few theorems comparing them.

Theorem 3.1.2. *The following implications hold:*
$$(X_n \xrightarrow{a.s.} X)\ or\ (X_n \xrightarrow{r} X) \Rightarrow (X_n \xrightarrow{P} X) \Rightarrow (X_n \xrightarrow{D} X)$$

for any $r \geq 1$*. If* $r > s \geq 1$*, then*

$$(X_n \xrightarrow{r} X) \Rightarrow (X_n \xrightarrow{s} X).$$

No other implications generally hold.

As this theorem implies, convergence in distribution is the most general convergence and holds in the most cases. As seen later the symmetric random walk converges in distribution to the Brownian motion, and a persistent random walk converges in distribution to the fractional Brownian motion.

Theorem 3.1.3. *The following conditions hold.*

1. *If* $X_n \xrightarrow{D} c$*, where* c *is constant, then* $X_n \xrightarrow{P} c$*.*
2. *If* $X_n \xrightarrow{P} X$ *and* $\mathbb{P}(|X_n| \leq k) = 1$ *for all* n *and some* k*, then* $X_n \xrightarrow{r} X$ *for all* $r \geq 1$*.*
3. *If* $P_n(\epsilon) = \mathbb{P}(|X_n - X| > \epsilon)$ *satisfies* $\sum_n P_n(\epsilon) < \infty$ *for all* $\epsilon > 0$*, then* $X_n \xrightarrow{a.s.} X$*.*

The following lemmas serve as proofs for these theorems and some more general results.

Lemma 3.1.4. *If* $X_n \xrightarrow{P} X$ *then* $X_n \xrightarrow{D} X$*. The converse is false.*

Proof. Suppose that $X_n \xrightarrow{P} X$ and write

$$F_n(x) = \mathbb{P}(X_n \leq x),\ F(x) = \mathbb{P}(x \leq x)$$

for the distribution functions of X_n and X respectively. If $\epsilon > 0$,

$$F_n(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X_n \leq x,\ X \leq x{+}\epsilon){+}\mathbb{P}(X_n \leq x,\ X > x{+}\epsilon) \leq F(x{+}\epsilon){+}\mathbb{P}(|X_n{-}X| > \epsilon).$$

Similarly,

$$F(x{-}\epsilon) = \mathbb{P}(X \leq x{-}\epsilon) = \mathbb{P}(X \leq x{-}\epsilon, X_n \leq x){+}\mathbb{P}(x \leq x{-}\epsilon, X_n > x) \leq F_n(x){+}\mathbb{P}(|X_n{-}X| > \epsilon).$$

Thus

$$F(x - \epsilon) - \mathbb{P}(|X_n - X| > \epsilon) \leq F_n(x) \leq F(x + \epsilon) + \mathbb{P}(|X_n - X| > \epsilon)$$

for all $\epsilon > 0$. If F is continuous at x then

$$F(x-\epsilon) \uparrow F(x) \text{ and } F(x+\epsilon) \downarrow F(x) \text{ as } \epsilon \downarrow 0.$$

Therefore $X_n \xrightarrow{D} X$. □

Lemma 3.1.5. *We have:*

1. *If $r > s \geq 1$ and $X_n \xrightarrow{r} X$ then $X_n \xrightarrow{s} X$.*
2. *If $X_n \xrightarrow{1} X$ then $X_n \xrightarrow{P} X$.*

Lemma 3.1.6. ***Markov's inequality***. *If X is any random variable, and $|X|$ has finite mean, then*

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}|X|}{a} \text{ for any } a > 0.$$

Theorem 3.1.7. *If $X_n \xrightarrow{D} X$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous then $g(X_n) \xrightarrow{D} g(X)$.*

Below is a diagram on how each of the modes of convergence interact. It illustrates which modes of convergence imply other ones following the directions of the arrows.

$$\begin{tikzcd}
a.s. \arrow[d] \arrow[dr] & r\text{th} \arrow[d] \arrow[dl] \\
P \arrow[r] & D
\end{tikzcd}$$

We write a.s for almost surely convergence, rth is in the rth mean, P is in Probability, and D is in Distribution. This diagram is the result of the above lemmas and proofs.

3.2 The Central Limit Theorem

In order to understand how the random walk that is the discretized version of Brownian motion in its limit has a normal distribution, it is imperative to know this next important theorem.

Theorem 3.2.1. ***The Central Limit Theorem (CLT)***. *Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite mean μ*

and finite non-zero variance σ^2*, and let* $S_n = \sum_{i=1}^{n} X_i$*. Then*

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \text{ as } n \to \infty,$$

where N(0,1) is the normal distribution with $\mu = 0$ *and* $\sigma = 1$*.*

It is useful to note that when the random variable X has $\mu = 0$ and $\sigma^2 = 1$ then

$$c\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \to N(0, c^2).$$

4 Stochastic Processes

Now that we understand what a random variable is, we need to look at a collection of them. Brownian motion and fractional Brownian motion are two stochastic processes that in this paper are used in getting the Black-Scholes formula and applying it to the stock market.

4.1 Background

Definition 4.1.1. A *stochastic process* X is a collection of random variables

$$(X_t, t \in T) = (X_t(\omega),\ t \in T,\ \omega \in \Omega),$$

defined on some space Ω and indexed by the set T. T is usually a subset of $\mathbb{R}$ and represents a time interval.

Definition 4.1.2. A stochastic process X is a function of two variables. For a fixed instant of time t, it is a random variable:

$$X_t = X_t(\omega),\ \omega \in \Omega.$$

For a fixed random outcome $\omega \in \Omega$, it is a function of time:

$$X_t = X_t(\omega),\ t \in T.$$

This function is called a *realization*, a *trajectory* or a *sample path* of the process X.

We can categorize stochastic processes in similar ways to random variables. Discrete random processes have discrete indexing sets T, for example, $T = \{0, \pm 1, \pm 2, ...\}$. Continuous random processes have indexing sets that are intervals, such as $T = [a, b]$ or $T = [a, \infty)$. It is also important to describe these processes in some fashion. Similar to random variables we look at their distribution, expectation, and describe their dependence structure.

Definition 4.1.3. The *finite−dimensional distributions* (*fidis*) of the stochastic process X are the distributions of the finite-dimensional vectors

$$(X_{t_1}, ..., X_{t_n}),\ t_1, ..., t_n \in T$$

for all possible choices of times $t_1, ..., t_n \in T$ and every $n \geq 1$.

The fidis completely determine the distribution of X. The collection of fidis is referred to as the *distribution of the stochastic process.*

Definition 4.1.4. Let $X = (X_t, t \in T)$ be a stochastic process and let $T \subset \mathbb{R}$ be an interval. Then X is said to have *stationary increments* if

$$X_t - X_s = X_{t+h} - X_{s+h} \text{ for all } t, s \in T \text{ and } h \text{ with } t + h, s + h \in T.$$

Definition 4.1.5. The stochastic process X has *independent increments* if for every choice of $t_i \in T$ with $t_1 < \cdots < t_n$ and $n \geq 1$,

$$X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$$

are independent random variables.

Definition 4.1.6. A stochastic process X is *Gaussian* if, for any positive integer n and any collection $t_1, ..., t_n \in T$, the distribution of the components of the random vector $(X_{t_1}, ..., X_{t_n})$ is normal.

Definition 4.1.7. The process X is a *Markov chain* if it satisfies the *Markov condition*

$$\mathbb{P}(X_n = s | X_0 = x_0, X_1 = x_1, \cdots, X_{n-1} = x_{n-1}) = \mathbb{P}(X_n = s | X_{n-1} = x_{n-1})$$

for all $n \geq 1$ and all $s,\ x_1, \ldots, x_{n-1} \in S$.

Markov Processes are random processes where the future processes are independent of the past. Brownian motion is a Markov process and, as we will see, fractional Brownian motion is not because the future processes will depend on the past.

4.2 Martingales

The notion of a martingale is useful in defining the Itô integral. The Itô integral is constructed in such a way that it constitutes a martingale. A martingale is a fair game where the net winnings are evaluated by conditional expectations.

Definition 4.2.1. The collection $(\mathcal{F}_t, t \geq 0)$ of σ-fields on Ω is called a *filtration* if

$$\mathcal{F}_s \subset \mathcal{F}_t \text{ for all } 0 \leq s \leq t.$$

Thus a filtration is just an increasing stream of information. Next we link a filtration with a stochastic process. First we need one more definition.

Definition 4.2.2. For any function $X : \Omega \to \mathbb{R}$ we define $\sigma(X)$ to be the smallest sigma field $\mathcal{F}$ on X with the property that $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ is measurable.

Now we can define what it means for a random process to be adapted to a filtration.

Definition 4.2.3. The stochastic process $Y = (Y_t, t \geq 0)$ is said to be adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ if

$$\sigma(Y_t) \subset \mathcal{F}_t \ for \ all \ t \geq 0.$$

Definition 4.2.4. Let $(\mathcal{F}_n, n \geq 1)$ be a filtration on the probability space $(\Omega, \mathcal{F}, P)$. A discrete-time martingale on $(\Omega, \mathcal{F}, P, \mathcal{F}_n)$ is a sequence $(X_n, n \geq 1)$ of integrable random variables on $(\Omega, \mathcal{F}, P)$, adapted to the filtration $\mathcal{F}_n$ (hence X_n is $\mathcal{F}_n$ measurable for all n) and

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$$

for all $n \geq 1$.

Lemma 4.2.5. *Let X_n be a martingale on some probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_n$, then*

$$\mathbb{E}(X_n) = \mathbb{E}(X_m)$$

for all n and m.

These are the basic discrete-time martingales, but what will become important for later are continuous-time martingales which are defined as follows.

Definition 4.2.6. Let $\mathcal{F}_t$ $(t \in I)$ denote a filtration on $(\Omega, \mathcal{F}, P)$ with an interval I of real numbers and let X_t $(t \in I)$ denote a set of integrable random variables on the probability space adapted to the filtration. Then X_t is a continuous-time martingale if

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s \text{ for all } s, t \in I, \ s \leq t.$$

This next proposition includes some martingales that will be used later in the paper to help model the stochastic behavior of stock prices. It shows us that Brownian motion is a martingale, and is used in calculating the expected value of geometric Brownian motion.

Proposition 4.2.7. *Let X_t denote a collection of random variables on $(\Omega, \mathcal{F}, P)$, and for $t \geq 0$ let $\mathcal{F}_t$ denote the σ-field generated by X_s. Suppose X_t and $X_t - X_s$ are $N(0,t)$ and $N(0, t-s)$ distributed random variables respectively and $X_t - X_s$ and X_r are independent for all $r, s, t, 0 \leq r \leq s \leq t$. The following hold.*

1. *W_t and $(X_t^2 - t)$ are martingales.*

2. *If μ and σ are real numbers, then $(e^{\mu t + \sigma X_t})$ is a martingale if and only if $\mu = \frac{-\sigma^2}{2}$.*

3. *If $\gamma > 0$, $Y_t = e^{-\frac{1}{2}\gamma^2 t + \gamma X_t}$ for $t \geq 0$ and $P_\gamma(A) = \int_A Y_\gamma dP$ when $A \in \mathcal{F}$, then $(\Omega, \mathcal{F}, P_\gamma)$ is a probability space. If $\mathcal{G}$ is a $\sigma - field$ on Ω with $\mathcal{G} \subset \mathcal{F}$ and Z is an integrable random variable on $(\Omega, \mathcal{F}, P_\gamma)$, then $\mathbb{E}(Z|\mathcal{G}) = \mathbb{E}(Y_\gamma Z)$. If $0 \leq s \leq t \leq \gamma$ and Z is $\mathcal{F}_t$ measurable, then $\mathbb{E}(Z|\mathcal{F}_s) = \mathbb{E}(Y_T Z|\mathcal{F}_s)$.*

5 Construction and Properties of Brownian Motion

Brownian motion is an important process that is critical to studying stochastic processes, specifically related to finance. To better understand Brownian motion and how it applies, it is useful to first look at how Brownian motion is constructed as a limit of random walks (see [9]). Afterwards a formal definition will be provided. Recall the simple random walk from earlier. We now define it further as a symmetric simple random walk where the random variable X can take the value of either 1 or -1 with $p = \frac{1}{2}$ and $q = \frac{1}{2}$ respectively. The particle following this random walk begins at $X_0 = 0$. This gives us

$$S_n = \sum_{i=1}^{n} X_i.$$

First let us look at some quick properties of this sum. S_n is the position of this particle at time n moving along the real number line. Given the probabilities we can see that $\mathbb{E}(S_n) = 0$ and that $\text{var}(S_n) = n, \ \ n \geq 0$. Currently it is a discrete random walk with steps taken every time of size 1 up to n. Now we need to "normalize" our random walk in such a way that when we take its limit we get Brownian motion. Consider a large integer $k > 1$. If we take a step every time at $1/k$ and make the step size $1/\sqrt{k}$ then at time t the particle will have taken $n = tk$ steps, a large number. Its new position will be

$$B_k(t) = \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} X_i.$$

The integer k is how we normalize the random walk. This will give us useful properties when we take the limit of the walk. We will prove that if we take the limit of this as $k \to \infty$ we will obtain the Brownian motion, a stochastic process defined below.

Definition 5.0.8. A stochastic process $B = (B_t, t \in [0, \infty))$ is called *Brownian motion* or a *Wiener process* if the following conditions are satisfied:

1. It starts at zero: $B_0 = 0$.

2. It has independent increments, meaning for every choice of $t_i \in T$ with $t_1 < \ldots < t_n$ and $n \geq 1$,

$$X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$$

are independent random variables.

3. For every $t>0$, B_t has a normal $N(0,t)$ distribution.

4. It has continuous sample paths, no jumps.

We see that Brownian motion is the limit of the random walk we defined above. The walk starts at $X_0 = 0$ as does Brownian motion. For independent increments take two non-overlapping time intervals (t_1, t_2) and (t_3, t_4) where $0 < t_1 < t_2 < t_3 < t_4$ are increasing. The corresponding increments

$$B_k(t_4) - B_k(t_3) = \frac{1}{\sqrt{k}} \sum_{i=t_3 k}^{t_4 k} X_i$$

$$B_k(t_2) - B_k(t_1) = \frac{1}{\sqrt{k}} \sum_{i=t_1 k}^{t_2 k} X_i$$

are independent because they are constructed from different X_i. Thus the limit as $k \to \infty$ will also have independent increments.
Next we can can look at the expected value and variance of Brownian Motion. Since the sum that we took the limit of had expected value 0, we can observe $\mathbb{E}(B_k(t)) = 0$. As noted above the variance of $S_n = n$, so here we just substitute in what n is, and we get $\text{var}(B_k(t)) = \frac{tk}{k} = t$ as $k \to \infty$ since $n = tk$ and our step size is $1/\sqrt{k}$. Plugging these values into the Central Limit Theorem with $c = \sqrt{t}$ we get

$$\lim_{k\to\infty} \frac{\sqrt{t}}{\sqrt{tk}} \sum_{i=1}^{tk} X_i \to N(0,t),$$

satisfying the third property of Brownian motion. The final condition of continuous sample paths is self-evident because the discrete random walk is continuous, therefore so is its limit. Now we can conclude that Brownian motion is the limit of a discrete random walk. Next we'll examine a few properties of this stochastic process.

Definition 5.0.9. A stochastic process $(X_t, t \in [0,\infty))$ is *H-self-similar* for some $H>0$ if its fidis satisfy the condition

$$(T^H B_{t_1}, ..., T^H B_{t_n}) = (B_{Tt_1}, ..., B_{Tt_n})$$

for every $T>0$, any choice of $t_i \geq 0, i = 1, ..., n$, and $n \geq 1$.

This implies that any properly scaled patterns of sample paths of random walks in any time interval have a similar shape, but aren't identical.

Definition 5.0.10. Brownian motion is 0.5-self-similar, i.e.,

$$(T^{1/2}B_{t_1}, ..., T^{1/2}B_{t_n}) = (B_{Tt_1}, ..., B_{Tt_n})$$

for every $T>0$, any choice of $t_i \geq 0, i = 1, ..., n$, and $n \geq 1$

One of the reasons for stochastic calculus, which we will see later, is the fact that processes that follow Brownian motion are unbounded and non-differentiable causing classical integration methods to fail.

Definition 5.0.11. The process X_t has *Geometric Brownian Motion* if

$$X_t = e^{\mu t + \sigma B_t}, t \geq 0,$$

where B_t is Brownian motion.

This specific type of Brownian motion becomes important later as it is the process used in the Black-Scholes model. Therefore it is important to know the expectation and covariance of this process. Recall that for a random variable, Z, with distribution $N(0,1)$, that

$$\mathbb{E}e^{\lambda Z} = e^{\lambda^2/2}.$$

Proof.

$$\begin{aligned}\mathbb{E}e^{\lambda Z} &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{\lambda z} e^{-z^2/2} dz \\ &= e^{\lambda^2/2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-(z-\lambda)^2/2} dz \\ &= e^{\lambda^2/2}.\end{aligned}$$

□

From this fact, and the 0.5 self-similarity of Brownian motion that we described earlier, it follows that:

$$\mu_X(t) = e^{\mu t}\mathbb{E}e^{\sigma B_t} = e^{\mu t}\mathbb{E}e^{\sigma t^{1/2} B_1} = e^{(\mu + 0.5\sigma^2)t}.$$

For $s \leq t, B_t - B_s$ and B_s are independent, and $B_t - B_s = B_{t-s}$. Therefore the covariance is:

$$\begin{aligned}
c_X(t,s) &= \mathbb{E}X_tX_s - \mathbb{E}X_t\mathbb{E}X_s \\
&= e^{\mu(t+s)}\mathbb{E}e^{\sigma(B_t+B_s)} - e^{(\mu+0.5\sigma^2)(t+s)} \\
&= e^{\mu(t+s)}\mathbb{E}e^{\sigma[(B_t-B_s)+2B_s]} - e^{(\mu+0.5\sigma^2)(t+s)} \\
&= e^{\mu(t+s)}\mathbb{E}e^{\sigma(B_t-B_s)}\mathbb{E}e^{2\sigma B_s} - e^{(\mu+0.5\sigma^2)(t+s)} \\
&= e^{(\mu+0.5\sigma^2)(t+s)}\left(e^{\sigma^2 s} - 1\right).
\end{aligned}$$

It follows that the variance function of geometric Brownian motion is

$$\sigma_X^2(t) = e^{(2\mu+\sigma^2)t}\left(e^{\sigma^2 t} - 1\right).$$

Below is a graph of showing one example of Brownian motion:

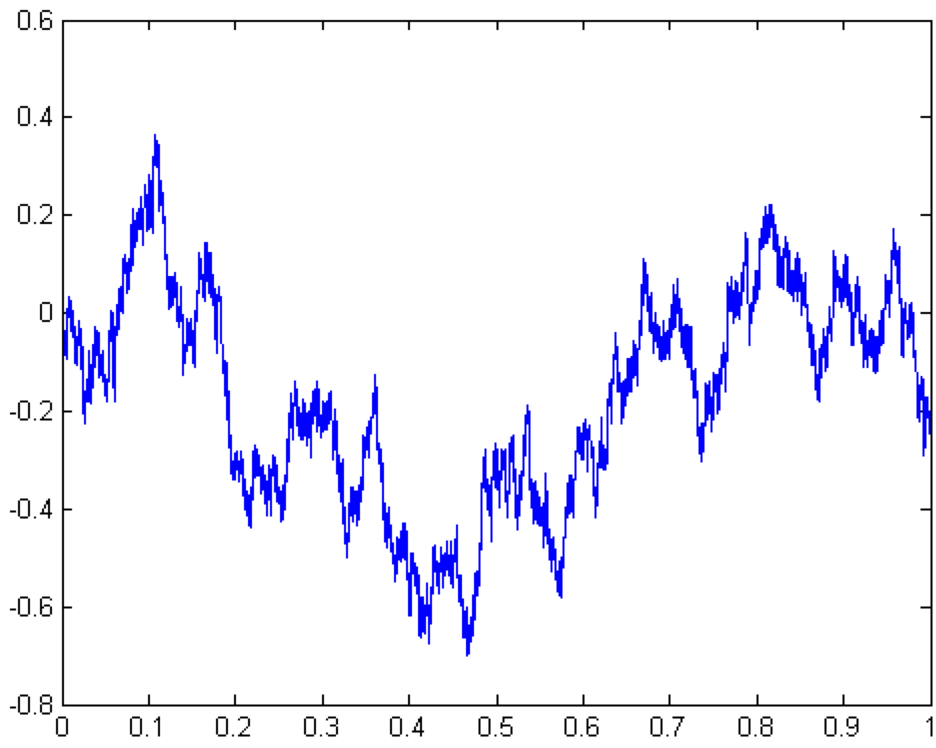

Figure 1: Sample random walk that approximates Brownian motion on the interval [0,1].

6 New Random Walk

After examining Brownian motion (Bm) and the random walk that it is constructed from, it is natural to put together our own version of a random walk with a twist. We construct and take the limit of a random walk that remembers the past.

Definition 6.0.12. Our random walk X_n^1 is defined by

$$X_0^1 = \{1 \text{ or } -1\} \text{ with probabilities } \left\{\frac{1}{2} \text{ and } \frac{1}{2}\right\}$$

and

$$X_n^1 = \{1 \text{ or } \ -1\} \text{ with probabilities} \left\{\frac{1}{2} + pX_{n-1}^1 \text{ and } \frac{1}{2} - pX_{n-1}^1\right\}$$

where $-\frac{1}{2} < p < \frac{1}{2}$.

We see that the first value X_0^1 has equal chance of being 1 or -1, but each successive X_i^1 depends on the previous value of X^1. Next, just as in the construction of Bm, we can look at the sum of this random walk and its properties, denoted R_n^1 :

$$R_n^1 = \sum_{i=1}^{n} X_n^1.$$

Just as with Brownian motion we want to look at this sums' properties. Here, though, it is a bit more complicated and not as straightforward.

Lemma 6.0.13. *The expected value of the random walk R_n^1 is 0.*

Proof. This is easily proved given our definition of X_0^1. We know that $E(X_0^1) = 0$ since it can take the value of 1 or -1 with equal probability. Thus

$$\mathbb{E}(R_n^1) = 1\left(\frac{1}{2} + (p\mathbb{E}(X_{n-1}^1))\right) - 1\left(\frac{1}{2} - (p\mathbb{E}(X_{n-1}^1))\right)$$

$$+ \cdots + 1\left(\frac{1}{2} + (p\mathbb{E}(X_1^1))\right) - 1\left(\frac{1}{2} - (p\mathbb{E}(X_1^1))\right) + 1\frac{1}{2} - 1\frac{1}{2}$$

and we know that $\mathbb{E}(X_n^1) = 0$ because each successive X value relies on the previous X value, all going back to X_0^1 which has expected value 0, making the expected value of all $X_n^1 = 0$. Thus we can conclude that $\mathbb{E}(R_n^1) = 0$. □

So far this looks like the Brownian motion, which also has expected value 0. Next we'll look at the variance of this sum.

Lemma 6.0.14. *The random walk R_n^1 has variance $n + 2\sum_{q=1}^{n}(2p)^q(n-q)$ for $n \geq 2$.*

Proof. We will prove this by induction.
Base Case: Let $n = 2$, then $\text{var}(R_2^1) = 2 + 2[(2p)(2-1) + 0] = 2 + 4p$ which holds. Then let $n = 3$, then $\text{var}(R_3^1) = 3 + 2[(2p)(3-1) + (2p)^2(3-2) + 0] = 3 + 8p + 8p^2$ which also holds.
Induction Step: Assume this holds for $n-1$ and n, we will prove for $n+1$.
Proof: Lets look at the difference between n and $n-1$.

$$\text{var}(R_n^1) - \text{var}(R_{n-1}^1) = n + 2\sum_{q=1}^{n}(2p)^q(n-q) - [n - 1 + 2\sum_{q=1}^{n-1}(2p)^q(n-1-q).$$

We see that all the terms in these series cancel out, giving us

$$\text{var}(R_n^1) - \text{var}(R_{n-1}^1) = 1 + 2\sum_{q=1}^{n-1}(2p)^q.$$

Next if we add this to the variance at step n and letting the n in the distance between the two be $n+1$ we get

$$\begin{aligned}\text{var}(R_n^1) + 1 + 2\sum_{q=1}^{n-1} &= n + 2\sum_{q=1}^{n}(2p)^q(n-q) + 1 + 2\sum_{q=1}^{n}(2p)^q \\ &= n + 1 + 2\sum_{q=1}^{n+1}(2p)^q(n+1-q) = \text{var}(R_{n+1}^1).\end{aligned}$$

This gives us exactly what we desired. $\square$

The variance of this new random walk is different than Brownian motion whose variance is just n. Here the variance depends on the value of p which makes intuitive sense because the larger p is, the farther apart the sum can get in either the positive or negative direction since the random walk has a higher probability to follow previous values, leading to a higher variance.

As with Brownian motion, we want to try and normalize this random walk by some variable, and then see what its limit is and if there are any similar properties. Similar to Brownian motion we take a step every time at $1/k$, and we make the step size $1/\sqrt{k}$. This is the same way that we normalized the simple random walk to achieve Brownian motion. The new position of this walk is now

$$R_k^1(t) = \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} X_i^1.$$

We then take the limit $k \to \infty$ as we did with Brownian motion, and examine what its expected value and variance now are. Let

$$R_t^1 = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{tk} X_i^1.$$

Lemma 6.0.15. *The stochastic process R_t^1 has expected value* 0 *and variance* $t\frac{1+2p}{1-2p}$.

Proof. The expected value of the sum R_n^1 is 0, therefore $\mathbb{E}(R_k^1(t)) = 0$, and thus in the limit as k approaches ∞, $\mathbb{E}(R_t^1) = 0$. We know from the previous lemma (6.0.14) that $\mathrm{var}(R_n^1) = n + 2\sum_{q=1}^{n}(2p)^q(n-q)$. Substituting in $n = tk$ and with the step size of $1/\sqrt{k}$ we see that

$$\begin{aligned}
\mathrm{var}(R_k^1(t)) &= \frac{1}{k}[tk + 2\sum_{q=1}^{tk}(2p)^q(tk-q)] \\
&= t + \frac{2}{k}\sum_{q=1}^{tk}(2p)^q(tk-q).
\end{aligned}$$

Now if we take the limit of this term as k goes to infinity we get

$$\mathrm{var}(R_k^1) = t + 2\sum_{q=1}^{\infty}(2p)^q t.$$

Using the formula for the sum of a geometric series and the fact that $2p < 1$, we get

$$\mathrm{var}(R_k^1) = t + 2t\frac{2p}{1-2p} = t\frac{1+2p}{1-2p}.$$

This is exactly what we desired. □

In addition to calculating the variance and expected value, there are two other properties of this stochastic process that are similar to Brownian motion.

Lemma 6.0.16. *The stochastic process R_t^1 has the following properties.*

1. *For every $t > 0$, R_t^1 has a normal $N(0,t)$ distribution.*

2. *R_t^1 has stationary increments, meaning*

$$R_t^1 - R_s^1 \sim R_{t-s}^1$$

for all $t, s \in T$, $s \leq t$.

Proof. To see that R_t^1 has a normal distribution we can look at the random walk from which it was constructed and apply the central limit theorem. We see as with Brownian motion, with $c = \sqrt{t}$, mean of 0, and $\sigma^2 = 1$, that

$$\lim_{k\to\infty} \frac{\sqrt{t}}{\sqrt{tk}} \sum_{i=1}^{tk} X_i^1 \to N(0,t).$$

Stationary increments follow from the normal distribution. $R_t^1 - R_s^1$ has the normal distribution of $N(0, t-s)$ which is the same distribution for R_{t-s}^1. Therefore they are distributed the same, meaning that this process has stationary increments. □

We see from this and the above properties that the differences between this stochastic process and Brownian motion are the variance and the fact that Brownian motion has independent increments while this process does not because of the persistence term. One can also consider history dependent random walks depending on 2 or more previous steps. We investigated numerically such random walks in the Appendix A.

7 Itô Calculus

7.1 The Riemann-Stieltjes Integral

Before going into the Itô integral, it is useful to have some quick background on the Riemann-Sieltjes integral and how Brownian motion is involved. We follow Mikosch's book on Stochastic calculus in deriving Itô 's lemma ([6]). Suppose for simplicity that f is a real-valued function defined on the interval $[0,1]$. Let

$$\tau_n : \quad 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1,$$

be a partition of the interval. Also define

$$\Delta_i g = t_i - t_{i-1}, \quad i = 1, \ldots, n.$$

This is the size of each interval. Then we can say that,

Definition 7.1.1. If the limit

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(y_i)\Delta_i g$$

exists and S is independent of the choice of the partitions of the Riemann integral, then S is called the $Riemann-Stieltjes\ integral\ of\ f\ with\ respect\ to\ g\ on$ $[0,1]$. We can write

$$S = \int_0^1 f(t)dg(t).$$

Definition 7.1.2. The real-valued function h on [0,1] is said to have *bounded p-variation* for some $p > 0$ if

$$\sup \sum_{t=1}^{n} |h(t_i) - h(t_{i-1})|^p < \infty$$

where the supremum is taken over all the partitions $0 = t_0 {<} t_1 {<} \cdots {<} t_{n-1} {<} t_n = 1$ of the interval $[0,1]$.

Definition 7.1.3. The Reimann-Stieltjes integral $\int_0^1 f(t)dg(t)$ exists if the following conditions are satisfied:

1. The functions f and g do not have discontinuities at the same point $t \in [0,1]$.

2. The function f has bounded p-variation and the function g has bounded q-variation for some $p > 0$ and $q > 0$ such that $p^{-1} + q^{-1} > 1$.

These become important because we can look at how it applies to Brownian motion. Let $B = (B_t, t \geq 0)$ denote Brownian motion. Assume f is a deterministic function or the sample path of a stochastic process. If f is differentiable with a bounded derivative on [0,1], then the Riemann-Stieltjes integral

$$\int_0^1 f(t) dB_t(\omega)$$

exists for every Brownian sample path $B_t(\omega)$.

7.2 The Itô Integral

Let us start with the usual Brownian motion, denoted $B = (B_t, t \geq 0)$ and its filtration of $\mathcal{F}_t = \sigma(B_s, s \leq t),\ t \geq 0$. To start with, we look at the Itô stochastic integral for "simple" processes, or processes whose paths assume only a finite number of values. These are defined as:

Definition 7.2.1. The stochastic process $C = (C_t, t \in [0, T])$ is said to be *simple* if it satisfies the following properties:

1. There exists a partition

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

   and a sequence of random variables X_i such that

$$C_t = \begin{cases} X_n, \text{ if } t = T, \\ X_i, \text{ if } t_{i-1} \leq t < t_i,\ i = 1, \ldots, n. \end{cases}$$

2. The sequence X_i is adapted to $\mathcal{F}_{t_{i-1}}$, that is X_i is a function of Brownian motion up to a time t_{i-1}.

Definition 7.2.2. The *Itô stochastic integral of a simple process* C on $[0, T]$ is given by

$$\int_0^T C_s dB_s = \sum_{i=1}^n C_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}) = \sum_{i=1}^n X_i \Delta_i B.$$

7.3 The Itô Lemma

Now that we know how to define an Itô integral we can look at how to actually calculate them using the Itô lemma. Let us start with a version of Taylor's formula with remainder. If $f(x)$ is an $(n+1)$ times differentiable function on an open interval containing x_0 then the function can be written as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0)^{n+1},$$

where θ is some point between x_0 and x.

We denote the last term of the previous formula by R_{n+1}. Suppose we have a function $F(y, z)$ and it has partial derivatives up to order three on an open disk containing the point with coordinates (y_0, z_0). Define $f(x) = F(y_0 + xh, z_0 + xk)$ where h and k are small enough that they lie within a disk surrounding (y_0, z_0). If we apply Taylor's formula with $x_0 = 0$ and $x = 1$ we get

$$f(1) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + R_3.$$

When we differentiate $f(x)$ setting $x = 0$ we also get

$$\begin{aligned} f'(0) &= hF_y(y_0, z_0) + kF_z(y_0, z_0) \\ f''(0) &= h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0), \end{aligned}$$

under the assumption that $F_{yz} = F_{zy}$. Now if we substitute these two equations into the first one we get

$$\begin{aligned} \Delta F &= f(1) - f(0) = F(y_0 + h, z_0 + k) - F(y_0, z_0) \\ &= hF_y(y_0, z_0) + kF_z(y_0, z_0) + \frac{1}{2}(h^2 F_{yy}(y_0, z_0) + 2hkF_{yz}(y_0, z_0) + k^2 F_{zz}(y_0, z_0)) + R_3. \end{aligned}$$

We can use this equation to derive Itô's lemma. As before let X be a random variable described by an Itô process of form

$$dX = a(X, t)dt + b(x, t)dW(t),$$

where $dW(t)$ is a normal random variable. Let $Y = F(X,t)$ be another random variable described by X and t. Now using the Taylor series expansion for Y we get

$$\Delta Y = F_X \Delta X + F_t \Delta t + \frac{1}{2} F_{XX}(\Delta X)^2 + F_{Xt}\Delta X \Delta t + \frac{1}{2} F_{tt}(\Delta t)^2 + R_3$$

$$= F_X(a\Delta t + bdW(t)) + F_t\Delta t + \frac{1}{2}F_{XX}(a\Delta t + bdW(t))^2 + F_{Xt}(a\Delta t + bdW(t))\Delta t + \frac{1}{2}F_{tt}(\Delta t)^2 + R_3.$$

We replaced ΔX with the discrete version of its Itô process. The remainder term R_3 contains terms of $(\Delta t)^k$ where $k \geq 2$, thus as Δt becomes small we get

$$\Delta Y \approx F_X(a\Delta t + bdW(t)) + F_t\Delta t + \frac{1}{2}F_{XX}b^2(dW(t))^2,$$

where $dW(t)$ is a normal random variable with expected value zero and variance Δt. Then

$$\text{var}(dW(t)) = \mathbb{E}[(dW(t))^2] - \mathbb{E}[dW(t)]^2,$$

which implies that $\mathbb{E}[(dW(t))^2] = \Delta t$, which we can use as an approximation of $(dW(t))^2$. Finally we obtain

$$\Delta Y \approx F_X(a\Delta t + bdW(t)) + F_t\Delta t + \frac{1}{2}F_{XX}b^2\Delta t.$$

In summary,

Lemma 7.3.1. *Itô's Lemma. Suppose that the random variable X is described by the Itô process*

$$dX = a(X,t)dt + b(X,t)dW(t),$$

where $dW(t)$ is a normal random variable. Suppose the random variable $Y = F(X,t)$. Then Y is described by the following Itô process:

$$dY = \left(a(X,t)F_X + F_t + \frac{1}{2}(b(X,t))^2F_{XX}\right)dt + b(X,t)F_X dW(t).$$

Now that we have the tools to differentiate and integrate normal random variables we can apply this to the stochastic process of stock prices to achieve the Black-Scholes Equation.

8 The Black-Scholes Option Pricing Formula

8.1 Motivation

Our ultimate end goal is the Black-Scholes formula for pricing options. In order to study this equation and how to more accurately estimate it, we need to really understand where it comes from, what it is doing, and what variables are involved and why. Let us start off with a quick background in mathematical finance and options.

Assume that the price X_t of a risky asset, for example, a stock at time t is given by a geometric Brownian motion of the form

$$X_t = f(t, B_t) = X_0 e^{(c-.5\sigma^2)t+\sigma B_t},$$

where $B = (B_t, t \geq 0)$ is a Brownian motion. This assumption on X comes from the fact that X is a unique "strong" solution ([6]) of the linear stochastic differential equation

$$X_t = X_0 + c\int_0^t X_s ds + \sigma\int_0^t X_s dB_s,$$

which we can formally write as

$$dX_t = cX_t dt + \sigma X_t dB_t.$$

This is believed to be a reasonable, though crude, first approximation of the real price process. Now let us assume we have a non-risky asset, in finance called a bond. We can also assume that a principal investment of β_0 in a bond yields

$$\beta_t = \beta_0 e^{rt}.$$

at time t. The principal β_0 is being continuously compounded with an interest rate r. In a financial portfolio one wants to hold a certain amount of shares in stock, a_t, and in bonds, b_t. We assume that a_t and b_t are stochastic processes adapted to Brownian motion, and that the pair is called a trading strategy. Note that the value of the portfolio, V_t, at time t is given by

$$V_t = a_t X_t + b_t \beta_t.$$

Then if we formulate this equation in terms of differentials we get

$$dV_t = d(a_t X_t + b_t \beta_t) = a_t dX_t + b_t d\beta_t,$$

which we can interpret using Itô's Lemma as

$$V_t - V_0 = \int_0^t d(a_s X_s + b_s \beta_s) = \int_0^t a_s dX_s + \int_0^t b_s d\beta_s.$$

This says that the value V_t of the portfolio at time t is equal to the initial investment V_0 plus the gain from stock and bond up to time t.

What this is all saying in non-mathematical language is that there are two types of investments that we are assume can be made: bonds and stocks. Bonds are unique because they essentially guarantee a rate paid out, which we will call the risk free interest rate. It is called risk free because an investor will receive the given rate on a bond at its expiration date with 100 percent certainty. They are important to keep in mind because an investor would not be willing to invest in a riskier stock or option if the expected payout is less than the risk free, guaranteed payout of a bond. Stocks, on the other hand, are much harder to predict the payouts of. They are represented by a random variable because the value they take at a given time in the day is unpredictable. Each stock has a different level of risk or volatility depending on numerous factors such as the companies past performance, current news, public opinion, pricing on the stock, the market the company is in, etc.

However, on average, we can think of a stock mathematically as following some type of Brownian motion, in either the upward or downward trajectory. What is important to keep in mind when looking at stocks is, besides all the news and public opinion, the volatility of the stock and the current price of the stock. Given our simple assumption of a world with only these two types of investments, an investor can then split his/her money between stocks and bonds to try and achieve the optimal rate of return while minimizing his/her risk.

Beyond stocks and bonds we get into a more nuanced area of finance: options. These were the main focus of Black and Scholes because their goal was to put an accurate price on these options by modeling them mathematically. First we will start with the bare bones, what an option is.

8.2 Options

Above we discussed what stocks are. When an investor chooses to purchase a stock, they usually pay the current price of the stock on the market. Here we represent the stock's current price as S. An investor will buy the stock at this price, then in theory when the stock's price rises above what the investor initially paid, they will sell it and make a profit. What makes options interesting is that they give an investor the option to buy or sell the stock in the future at a given price, called the strike price, or K.

There are two types of options, calls and puts. A call is when an investor locks in the strike price K, and in the future at whenever the option expires has the opportunity to buy the stock at the strike price, no matter how much the stock is worth. The investor can then sell the stock at the given price, and make a profit of $S - K$. The investor is paying a premium in order to buy the stock at a certain locked in price in the future, which means they are betting on the stock rising in value.

The second type of option, called a put, is the opposite of a call. The investor purchases the option at a certain strike price, K, which at the expiration of the option they get the option to sell it at the strike price, once again making a profit of $|K - S|$ since they purchase the stock at price S. Here though, the investor is betting that the value of the stock will decrease below the strike price, otherwise they won't make a profit. In addition to puts and calls, there are two types of options, an American or a European option. A European option can only be exercised at the expiration date of the option, while an American option can be exercised any time over the life of the option. American options, mathematically, are much harder to model since there is no discrete expiration time, so we focus only on European options here.

The strike price K determines whether the option is either in the money or out of the money. An in the money option is one where the strike price is less than the current stock price. An out of the money option is one where the strike price is greater than the current stock price. In the money calls cost more than out of the money calls because they could be exercised right away to make a profit, whereas in order to execute and make a profit on an out of the money call the stock price must rise above the strike price.

We've talked about these option as something that one can purchase and exercise after the expiration date. The main question is, how do we determine what the price of an option is? That is what Black-Scholes attempts to do. Before looking at their complicated formula let us start off with a simpler example. Intuitively the option should be priced at such a value that eliminates arbitrage. Arbitrage exists when two financial investments are mis-priced relative to each other, such that it is possible to make money off of the mis-pricing without any associated risk. For an option, it can't be priced at such a value such that makes more money than the risk free interest rate. Arbitrage can lead us to the *Put-Call Parity Formula*. It is an arbitrage-free relationship between a European call and put with the same strike price and expiration date. Let P_e denote a European put, C_e denote a European call, r denote the risk-free interest rate, and T the expiration date. Then

$$P_e + S = C_e + Ke^{-rT}.$$

This will become important later in changing the Black-Scholes Call Pricing Formula to price a put. In addition it is useful to think about because it contains many elements that are key to pricing an option as we have discussed, the stock price, the strike price, the time of the option, and the risk free interest rate.

8.3 The Black-Scholes Partial Differential Equation

In this section we will derive the Partial Differential Equation, or PDE, that governs options pricing (see [1]). The next section will solve it giving us the famous options pricing formula. Let us start with the price of the underlying security or stock, S. Suppose it can be modeled by the stochastic process

$$dS = \left(\mu + \frac{\sigma^2}{2}\right) Sdt + \sigma S dW(t).$$

Before proceeding let us examine this process. A security or stock S has a stochastic part and a deterministic part. A stock follows a general trend on average based upon historical performance and the market, which is captured here by the dt portion. What affects this is the stocks volatility, its current price, and the mean value of the stock over time, μ. The stock also has a random part to it which as described earlier follows Brownian motion. Here this is represented by $dW(t)$. The randomness in a stock depends on the stocks current price and its volatility since it is not deterministic and can't be examined over long periods of time. Therefore when we look at the change in the stock price S, we need to take into account both the stochastic and deterministic nature of the stock.

Let $F(S,t)$ be the value of any option with time t and stock price S. We can apply Itô's Lemma to this to get the following equation:

$$dF = \left(\left[\mu + \frac{\sigma^2}{2}\right] SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt + \sigma S F_S dW(t).$$

Now suppose an investor has a portfolio P which is created by selling options and buying Δ units of the stock S. The value of the portfolio is now $P = F - \Delta S$. Differentiating both sides we can look at the stochastic process governing the portfolio:

$$\begin{aligned} dP &= d(F - \Delta S) = dF - \Delta dS \\ &= \left(\left[\mu + \frac{\sigma^2}{2}\right] SF_S + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt + \sigma S F_S dW(t) + \Delta\left(\left(\mu + \frac{\sigma^2}{2}\right) S dt + \sigma S dW(t)\right) \\ &= \left(\left[\mu \frac{\sigma^2}{2}\right] S[F_S - \Delta] + \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt + \sigma S(F_S - \Delta) dW(t). \end{aligned}$$

We see that some of the randomness in this last equation can be reduced if we assume that $\Delta = F_S$, which gets rid of the $dW(t)$ term. This doesn't completely eliminate randomness though because the value of the stock S is still stochastic. Under this simplification we get

$$dP = \left(\frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt.$$

Here we are assuming an arbitrage-free setting as discussed previously. Therefore the returns from investing in this portfolio should be the equal to investing the same amount of capital in risk-free bonds paying interest rate r. Thus

$$0 = rPdt - dP = rPdt - \left(\frac{1}{2}\sigma^2 S^2 F_{SS} + F_t\right) dt.$$

$$rP = \frac{1}{2}\sigma^2 S^2 F_{SS} + F_t.$$

Substituting in $P = F - \Delta S$ we get

$$rF = F_t + rSF_S + \frac{1}{2}\sigma^2 S^2 F_{SS}.$$

This final equation is our Black-Scholes PDE that we will solve to obtain the Call-Options Pricing Formula. A PDE can have many solutions, but we are interested in the one we get when we impose certain boundary conditions related to options.

8.4 The Black-Scholes Formula

After solving the Black-Scholes PDE, which we don't go into in this paper (see, for example, [6]), we finally get the famous *Black-Scholes European Call Option Pricing Formula*

$$C(S,t) = S\phi(d_1) - Ke^{-r(T-t)}\phi(d_2),$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and ϕ is the normal distribution function. We can see from this equation that everything we discussed earlier is included. The price of a European Call depends on the stock price S, the volatility of the stock σ, the risk-free interest rate r, the strike price of the option K, and the timeline of the option T. Although this famous equation is widely used in finance and in pricing options it is not without its flaws. One question often discussed is do stock prices truly follow Brownian motion? That is do stocks' past performance have no impact on its future? We can examine this issue by creating a new type of random walk that remembers the past, and introduce a parameter that calculates how much a stock truly remembers the past. Then we'll use this new random walk to get another type of Brownian motion, which in the end will give us another version of the Black-Scholes equation.

9 The Fractional Brownian Motion Black-Scholes Formula

9.1 The Hurst Parameter

The Hurst parameter, denoted H, is a statistical measure used to classify time series. The value of H varies between 0 and 1. In this instance we will be using it to classify financial time series. $H = .5$ means that the time series is completely random. An H between 0 and .5 indicates that the time series is anti-persistent. With such an H value, the series will tend to revert to the mean. That is, if the series is high in one time period, it is likely to do the opposite and be lower in the next time period. An $H > .5$ is persistent, meaning that if the time series is up during one period, it has a higher chance to keep going up. Essentially the series remembers the past and tends to follow it. Based upon multiple papers on the Hurst parameter in finance H is generally persistent and greater than .5, so this is what we will focus on.

This concept was developed in the 19th century by Harold Edwin Hurst when he observed trends in the flooding of the Nile River and how a low flood year tended to be followed by another low flood year, and so on. It is important and useful in finance because the value of H can be used to predict future stock prices. This isn't without issues though, evidence is mixed on whether financial time series such as stocks are long term memory processes. This concept conflicts with the Efficient Market Hypothesis that all information in the market is incorporated into the financial time series, thus already reflected in its price. An offshoot of the EMH is the Random Walk Hypothesis. The RWH is the assumption that financial time series follow a completely random walk and are unpredictable meaning that $H = .5$. Although opinions on the application of the Hurst parameter to finance are mixed, we are going to examine estimating it for certain stocks, using it to construct a different kind of Brownian motion, giving us a Black-Scholes equation that remembers the past, and compare this to the completely random ($H = .5$) Black-Scholes equation.

Further on we will go into how the Hurst parameter is used in Black-Scholes, but first we need to know how to mathematically calculate it. One of the problems with the Hurst parameter is that it is not so much calculated as it is estimated, and there are multiple ways to go about doing so. The simplest and most applicable way to calculate it based upon how we use the Hurst parameter is called the *Rescaled Range* (see [8]).

Definition 9.1.1. The *Rescaled Range* is a statistical measure of the variability of a time series, denoted R/S. It is given by

$$R/S = \frac{R(n)}{\sigma},$$

where $R(n)$ is the range of the time series and σ is the standard deviation.

Here is a step by step walkthrough to calculating R/S with the data broken down into time periods $t = 1, 2, ..., n$.

1. Calculate the mean value (here its the return of a stock) of the time series:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

where the X_i are the values of the time series at each period $1, 2, ..., n$.

2. Calculate each of the mean adjusted values

$$Y_t = X_t - \mu \qquad t = 1, 2, ..., n.$$

3. Calculate the cumulative deviate series

$$Z_t = \sum_{t=1}^{n} Y_t \qquad t = 1, 2, ..., n.$$

4. Calculate the range of the series

$$R_t = \max(Z_1, \ldots, Z_t) - \min(Z_1, \ldots, Z_t).$$

5. Calculate the standard deviation of the series

$$\sigma_t = \sqrt{\frac{1}{t} \sum_{i=1}^{t} (X_t - \mu)^2}.$$

6. Finally calculate the rescaled range

$$(R/S)_t = R_t/\sigma_t \qquad t = 1, 2, ..., n.$$

The rescaled range is calculated using these n data points in the time series. To get the Hurst parameter you plot $\log(R/S)$ against $\log(n)$ to get a point on the graph. Next you choose a larger n, so more data points in the time series, and recalculate R/S, then again plot $\log(R/S)$ against $\log(n)$ on the same graph. Keep repeating this process for multiple n and plotting on the graph. Finally plot the line of best fit on this graph and the slope of the line is the Hurst parameter. Theoretically this makes sense because an $H = .5$, or a slope of $\frac{1}{2}$ implies that there is no dependence on previous data since the rescaled range vs. the number of data points increases in a 1:1 ratio.

Now apply this to finance and stocks. If you sell a stock at time t at price P_t, and you purchased that stock at time $t-\Delta$ at price $P_{t-\Delta}$, then we can calculate the log return of the stock which removes most of the market trend.

Definition 9.1.2. The *log return* of a stock, denoted $re^{t-\Delta}$ is given by

$$re^{t-\Delta} = \log(P_t) - \log(P_{t-\Delta}).$$

Next you break the stock into blocks of set time periods to use as the n in calculating the Hurst parameter. For example, take the Apple stock with the ticker symbol AAPL. Calculate the average return of Apple in the past 5 days, then 10 days, then 15, and so on until you reach 500 days of past history. Here the $n's$ are $5, 10, 15, \ldots, 500$ generating 100 data points on the graph. For each of these n we go through the calculations above getting the rescaled range and plotting its log against $\log(n)$. The slope of this line will be an estimation of the Hurst parameter for the Apple stock.

9.2 Construction and Properties of Fractional Brownian Motion

We need to first construct a discrete random walk that involves the Hurst parameter and remembers the past, then take its limit (see [2]). If $0 < H < \frac{1}{2}$ we have to construct a different random walk than in the case of $H > \frac{1}{2}$, because when $H < \frac{1}{2}$ the time series is anti-persistent. As discussed above, the large portion of financial time series are persistent, so we can focus on a correlated random walk for when $H > \frac{1}{2}$.

Definition 9.2.1. For any $p \in [0,1]$ the *Correlated Random Walk* X^p with persistence p is a $\mathbb{Z}$-valued discrete process such that

1. $X_0^p = 0,\ P(X_1^p = -1) = \frac{1}{2},\ P(X_1^p = 1) = \frac{1}{2}$,
2. For any $n \geq 1, \epsilon_n^p = X_n^p - X_{n-1}^p = 1$ or -1,
3. For any $n \geq 1, P(\epsilon_{n+1}^p = \epsilon_n^p | \sigma(X_k^p, 0 \leq k \leq n)) = p$.

This correlated random walk remembers the past with the persistence value. Note that the persistence p is different from the Hurst parameter H. Here p is used to define the discrete random walk that we take the limit of to achieve fractional Brownian motion whereas H is the parameter used to measure the specific time series correlation. Before taking the limit of this walk we need a few more results presented below to further define it.

Proposition 9.2.2. *We have for any* $m \geq 1, n \geq 0, E[\epsilon_m^p \epsilon_{m+n}^p] = (2p-1)^n$.

Next we introduce some extra randomness into the persistence. Consider a probability measure μ on [0,1], we call P^μ the annealed law of the correlated walk associated with μ. This is the measure on $\mathbb{Z}^n$ defined by $P^\mu = \int_0^1 P^p d\mu(p)$. Now let $\epsilon_n^\mu = X_n^\mu - X_{n-1}^\mu$. This gives us:

Proposition 9.2.3. *For any* $m \geq 1, n \geq 0, E[\epsilon_m^\mu \epsilon_{m+n}^\mu] = \int_0^1 (2p-1)^n d\mu(p)$.

Now we are ready to take the limit of this correlated random walk. The idea is to come up with a probability measure μ such that the limit of the walk is a fractional Brownian motion ([2]).

Theorem 9.2.4. *Let* $H \in [\frac{1}{2}, 1]$. *Let* μ^H *be the probability on* $[\frac{1}{2}, 1]$ *with density* $(1-H)2^{3-2H}(1-p)^{1-2H}$. *Let* $X^{\mu^H,i}$ *be a sequence of independent processes of law* P^{μ^H},

$$\mathcal{L}^{\mathcal{D}} \lim_{N\to\infty} \mathcal{L} \lim_{M\to\infty} c_H \frac{X^{\mu^H,1} + \cdots + X^{\mu^H,M}}{N^H \sqrt{M}} = B_H(t),$$

with $c_H = \sqrt{H(2H-1)/\Gamma(3-2H)}$*.* $\mathcal{L}$ *means convergence in the sense of finite-dimensional distributions and* $\mathcal{L}^{\mathcal{D}}$ *means weak convergence in the Skorohod topology on* $D[0,1]$*, the space of cadlag functions on* $[0,1]$*.*

Proof. We omit the proof here, but see [2]. □

Definition 9.2.5. For $0 < H < 1$, *fractional Brownian motion* (fBm) is a stochastic process, B_t^H, which satisfies the following,

1. B_t^H is Gaussian, i.e. for every $t>0$, B_t^H has a normal distribution,
2. B_t^H is H-self-similar, that is $B_{at}^H \sim |a|^H B_t^H$,
3. It has stationary increments, $B_t^H - B_s^H \sim B_{t-s}^H$.

Lemma 9.2.6. *Fractional Brownian motion has the following properties:*

1. $B_0^H = 0$,
2. $E(B_t^H) = 0 \ \forall t \geq 0$,
3. $E((B_t^H)^2) = t^{2H} \ \forall t \geq 0$,
4. $\mathrm{cov}(B_t^H, B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.

Below are some sample graphs of fBm for different values of H.

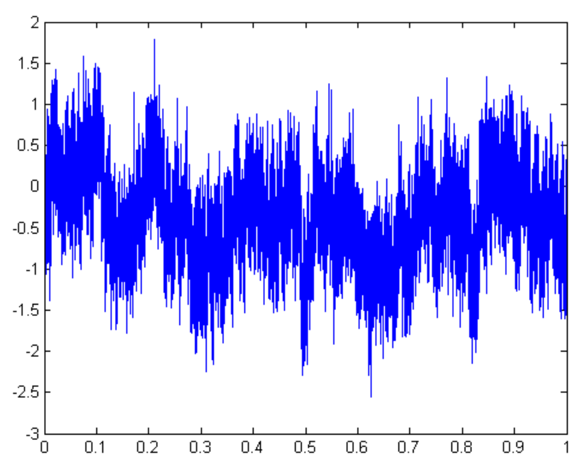

Figure 2: Sample random walk that approximates fractional Brownian motion for a Hurst parameter of .1 on the interval [0,1].

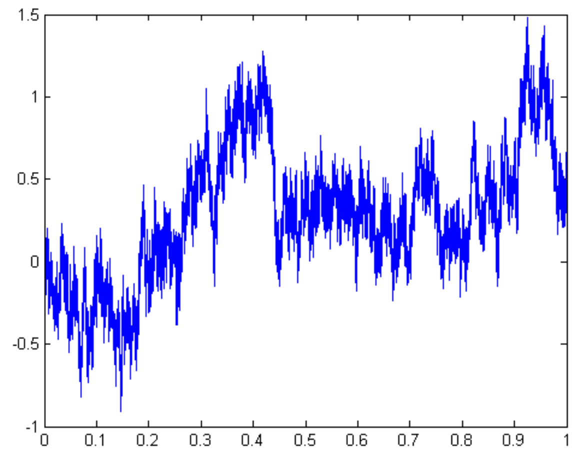

Figure 3: Sample random walk that approximates fractional Brownian motion for a Hurst parameter of .3 on the interval [0,1].

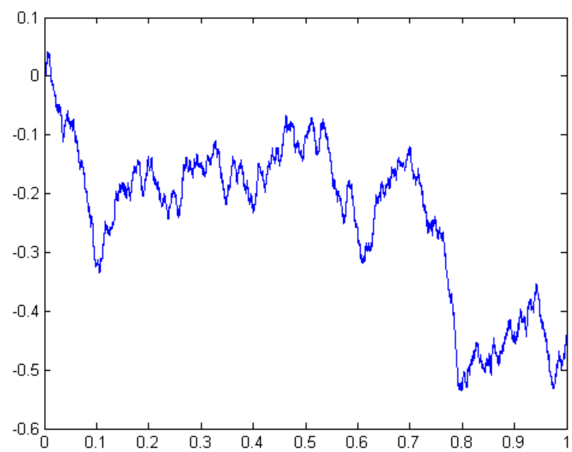

Figure 4: Sample random walk that approximates fractional Brownian motion for a Hurst parameter of .7 on the interval [0,1].

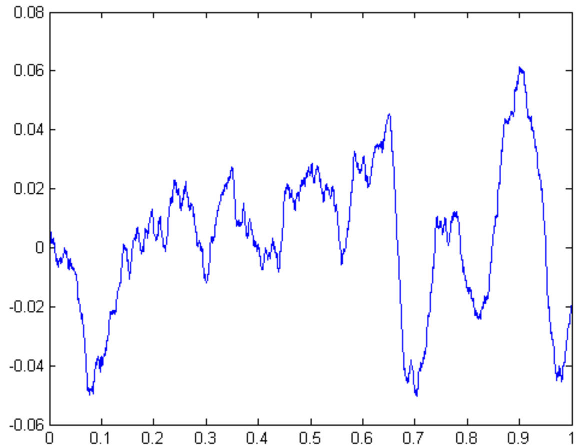

Figure 5: Sample random walk that approximates fractional Brownian motion for a Hurst parameter of .9 on the interval [0,1].

As with normal Brownian motion, we can examine a market consisting of a riskless asset or bond $A(t)$ with

$$dA(t) = rA(t)dt,$$

as well as as of a risky asset or stock $S(t)$ with

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_t^H.$$

The process in this equation is a geometric fractional Brownian motion and the parameter r and σ are assumed to be constant symbolizing the interest rate and the volatility respectively.

9.3 The Fractional Black-Scholes Model

We will not go into the details on how to derive the model in this paper as it is well studied (see, for example, [5]). The price of a fractional European call with strike price K and maturity T is given by

$$C(S,t) = S\phi(d_1^H) - Ke^{-r(T-t)}\phi(d_2^H),$$

where

$$\begin{aligned} d_1^H &= \frac{\ln(\frac{S}{K}) + r(T-t) + \frac{1}{2}\rho_H \sigma^2 (T-t)^{2H}}{\sqrt{\rho_H}\sigma(T-t)^H} \\ d_2^H &= d_1^H - \sqrt{\rho_H}\sigma(T-t)^H \\ \rho_H &= \frac{\sin(\pi(H-\frac{1}{2}))\Gamma(\frac{3}{2}-H)^2}{\pi(H-\frac{1}{2})\Gamma(2-2H)} \end{aligned}$$

and Γ is the gamma function.

9.4 The Relationship Between H and the Options Price

Now that we have the fractional Black-Scholes equation, we want to see how it differs mathematically from the standard Black-Scholes equation. To do this we can first note that the one difference between the two equations is the addition of the Hurst parameter H. We want to examine how the fractional model changes when H changes, and to do this we can take the derivative with respect to H.

$$\zeta = \frac{\partial C}{\partial H} = S\phi(d_1^H)\frac{\partial d_1^H}{\partial H} - Ke^{-r(T-t)}\phi(d_2^H)\frac{\partial d_2^H}{\partial H}.$$

Theorem 9.4.1. *The partial derivative of the fractional call price C with respect to the Hurst Parameter H is given by*

$$\zeta = S\phi(d_1^H)\sqrt{\rho_H}\sigma(T-t)^H \frac{(\psi(1-H) - \psi(H+\frac{1}{2}) + 2\ln 2 + 2\ln(T-t)}{2},$$

where $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

Proof. Most of this proof involves lots of algebra, but we sketch it out here.

$$\begin{aligned} \zeta = \frac{\partial C}{\partial H} &= S\phi(d_1^H)\frac{\partial d_1^H}{\partial H} - Ke^{-r(T-t)}\phi(d_2^H)\frac{\partial d_2^H}{\partial H} \\ &= S\phi(d_1^H)\frac{\partial(\sqrt{\rho_H}\sigma(T-t)^H)}{\partial H}. \end{aligned}$$

Before we can calculate this partial we need to find $\frac{\partial \rho_H}{\partial H}$. Recall that

$$\rho_H = \frac{\sin(\pi(H-\frac{1}{2}))\Gamma(\frac{3}{2}-H)^2}{\pi(H-\frac{1}{2})\Gamma(2-2H)}.$$

We first take the derivative of the top and of the bottom, then apply the quotient rule. Let $\frac{\partial T}{\partial H}$ be the derivative of the numerator and let $\frac{\partial B}{\partial H}$ be the derivative of the denominator. Then we get

$$\frac{\partial T}{\partial H} = \left(\Gamma\left(\frac{3}{2} - H\right)\right)^2 \sin\left(\pi\left(H - \frac{1}{2}\right)\right)\left[\pi\cot\left(\pi\left(H - \frac{1}{2}\right)\right) - 2\psi\left(\frac{3}{2} - H\right)\right]$$

and

$$\frac{\partial B}{\partial H} = \pi\Gamma(2 - 2H)\left(1 - 2\left(H - \frac{1}{2}\right)\psi(2 - 2H)\right).$$

Applying the quotient rule with these values yields

$$\frac{\partial \rho_H}{\partial H} = \rho_H\left[\pi\cot\left(\pi\left(H - \frac{1}{2}\right)\right) - 2\psi\left(\frac{3}{2} - H\right) - \frac{1}{H - \frac{1}{2}} + 2\psi(2 - 2H)\right].$$

Next we use the below properties of the digamma function,

$$\begin{aligned}
\psi(x+1) &= \psi(x) + \frac{1}{x} \\
\psi(1-x) - \psi(x) &= \pi\cot(\pi x) \\
\psi(2x) &= \frac{1}{2}\left(\psi(x) + \psi\left(x + \frac{1}{2}\right) + 2\ln 2\right),
\end{aligned}$$

to simplify the derivative to

$$\frac{\partial \rho_H}{\partial H} = \rho_H\left[\psi(1 - H) - \psi\left(H + \frac{1}{2}\right) + 2\ln 2\right].$$

Now that we have the derivative of ρ_H we can apply this to find $\frac{\partial(\sqrt{\rho_H}\sigma(T-t)^H)}{\partial H}$. First note that we have a square root of ρ_H in this, so let $r = \rho_H\sigma^2(T-t)^{2H}$, so that $\sqrt{r}$ is the value that we need to take the derivative of. Notice that

$$\frac{\partial\sqrt{r}}{\partial H} = \frac{1}{2}\frac{1}{\sqrt{r}}\frac{\partial r}{\partial H},$$

and that

$$\frac{\partial r}{\partial H} = \rho_H\sigma^2(T-t)^{2H}\left[2\ln(T-t) + \psi(1-H) - \psi(H + \frac{1}{2}) + 2\ln 2\right].$$

Combining these two equations and solving for our partial we get

$$\frac{\partial\sqrt{r}}{\partial H} = \sqrt{\rho_H}\sigma(T-t)^H \frac{(\psi(1-H) - \psi(H+\frac{1}{2}) + 2\ln 2 + 2\ln(T-t)}{2}$$

giving us the desired result. □

Since we are interested in when the stock remembers the past, mainly when $\frac{1}{2} < H < 1$, we want to see how this derivative changes for these H values. Besides H, the only value that affects the sign of the derivative is $(T-t)$ which we will call τ here. The following lemma provides two insights as to the sign of the derivative.

Lemma 9.4.2. *We restrict H to $\frac{1}{2} < H < 1$, and $\tau = (T-t) > 0$, then the following are true:*

1. *If $\tau \leq 1$, then $\forall\ \frac{1}{2} < H < 1$, $\frac{\partial C}{\partial H} < 0$.*
2. *If $\tau > 1$, then for some critical H value, call it H_c,*

$$\frac{\partial C}{\partial H_c} = 0$$

$$\frac{\partial C}{\partial H} > 0 \quad for \quad \frac{1}{2} < H < H_c$$
$$\frac{\partial C}{\partial H} < 0 \quad for \quad H_c < H < 1.$$

Proof. First we will begin by noting a couple things. Isolating and looking at the digamma terms, $\psi(1-H) - \psi(H+\frac{1}{2})$, and graph it for $\frac{1}{2} < H < 1$, it is always negative, but is the highest at $H = \frac{1}{2}$. If $H = \frac{1}{2}$, then we want to solve $\psi(1/2) - \psi(1)$. Using the properties of the digamma function, mainly $2\psi(2x) = \psi(x) + \psi(x+\frac{1}{2}) + 2\ln 2$, we can note $\psi(1/2) - \psi(1) = -2\ln 2$.

Next we can note that the term $S\phi(d_1^H)\sqrt{\rho_H}\sigma\tau^H > 0$ for any value of H and τ. The proof of the first statement then follows. At $H = \frac{1}{2}$, the $2\ln 2$ terms cancel, at at values of H greater than that, the digamma terms overwhelm the $2\ln 2$ term making it more negative. Next since $\tau \leq 1$, the term $2\ln\tau \leq 0$, therefore this entire term $(\psi(1-H) - \psi(H+\frac{1}{2}) + 2\ln 2 + 2\ln(T-t)$ is negative, thus $\frac{\partial C}{\partial H} < 0$. The proof of the second statement can be seen easily using the same logic from above, except this time the term $2\ln\tau > 0$, which is where the H_c value arises, and all H values less than that make the derivative positive, and all values of H greater than

(a) The change in call prices due to the Hurst parameter for $S = \$55$, $K = \$50$, $\sigma = 20\%$, $\tau = .5$, and $r = 2\%$.

(b) The change in call prices due to the Hurst parameter for $S = \$55$, $K = \$50$, $\sigma = 20\%$, $\tau = 1.5$, and $r = 2\%$.

Figure 6: Two different call option pricing paths due to the change in time of expiration.

the critical value are make the derivative negative as the digamma term overpowers the $2 \ln \tau$ term. $\square$

As we can see in the lemma, it is not only the value of the Hurst parameter that affects the option price, but the expiration date as well which makes economic sense. In the first instance when τ is less than 1, that is the option's expiration is less than a year. Then increasing the Hurst parameter or the amount the stock remembers the past, decreases the call price because on such a short time horizon, and the more the stock is dependent upon the past, the cheaper the call will become because it is more of a sure thing and no risk is priced in.

In the second instance, when $\tau > 0$ which represents longer call options, increasing H to a certain value will make the call more expensive depending on the expiration because, as with most investments, the longer the time horizon is, the riskier it becomes. Upon reaching a certain value though, the call price starts to decrease as H continues to increase because the dependence of the stock becomes more of a sure thing. These graphs showing the changing call price for two different expiration dates as H increases.

10 Applications to the Stock Market

10.1 Setup

Now that we have two different ways to calculate option pricing, one being the traditional model and the other being the fractional model, we can apply them using actual market data. Originally we chose three different stocks, Accenture (ACN), Apple (AAPL), and First Solar (FSLR), each of which has low volatility, medium volatility, and high volatility respectively. We started data collection on January 28, 2015. For each stock we recorded the current stock price as of that day, and the market options price for each stock at certain strike prices in weekly increments for a total of one month. Specifically the market options price on February 6th, February 13th, February 20th, and February 27th. Then using an excel spreadsheet with the Hurst parameter calculation, volatility calculation, the original Black-Scholes Formula, and the fBm Black-Scholes formula we estimated each of these stocks options prices going forward.

Next we recorded the actual stock price on each of the chosen days and calculated the percentage difference in the profit from each of the three estimates (actual option price, estimated Bm option price, fBm option price) and the difference in predicted stock price vs. actual stock price. After doing this it was clear that more than three data points were necessary so we went back using historic stock price data for seventeen more stocks (Microsoft, Capital One, American International Group, Sunopta, Exxon, Berkshire Hathaway A, Starbucks, Amazon, Target, Google, Blackberry, Netflix, BP, IBM, Southwest Airlines, Chipotle, and Adidas) and did the same analysis. The exception with these stocks is that historic option pricing data is only available at a high cost, thus we could only compare estimates between the two models without actual market option prices.

For each stock we took two values at each point in time, one where the strike price of the option was in the money and one where the strike price was out of the money.

10.2 Calculating Volatility

The one parameter of Black-Scholes that can vary in its calculation is the volatility, or σ. The calculation of this can set firms apart in trying to “beat the market” and get a better estimate of the option price of a stock. For the purposes of this paper we will use a standard, well-known calculation of a stock's volatility (see [4]). We start by taking the log-normal stock return day-to-day.

$$u_i = \ln\left(\frac{S_t}{S_{t-1}}\right).$$

After calculating that we can get the volatility using a simple formula.

$$\sigma = \frac{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}}{\sqrt{\tau}},$$

where $\bar{u}$ is the mean of all the u_i, τ is the length of the period in years, and n is the total number of observations. Although the stock market trades only 252 days out of the year, we used $\tau = 365$ as the time assumption because the Hurst parameter calculation and the Black-Scholes calculation use this as their time value.

10.3 Options Data

Detailed in the below tables is each stock's Cap Size (how large the company is), estimated Hurst parameter, and estimated volatility.

Table 1: Stocks Observed (1)

	ACN	*AAPL*	*FSLR*	*MSFT*	*COF*
CapSize (millions\$)	59,851	742,551	6,142	348,776	43,885
Hurst Parameter	0.357	0.622	0.694	0.472	0.450
Volatility	18.90%	26.74%	56.47%	22.32%	19.90%

Table 2: Stocks Observed (2)

	AIG	*STKL*	*XOM*	*BRK/A*	*SBUX*
CapSize (millions\$)	76,994	666	359,191	346,400	69,143
Hurst Parameter	0.534	0.594	0.492	0.527	0.494
Volatility	20.94%	41.30%	20.41%	15.87%	23.13%

Table 3: Stocks Observed (3)

	AMZN	*TGT*	*GOOGL*	*BBRY*	*NFLX*
CapSize (millions$)	175,797	50,046	387,172	5,214	26,960
Hurst Parameter	0.475	0.496	0.503	0.659	0.601
Volatility	40.07%	25.15%	25.78%	68.49%	52.70%

Table 4: Stocks Observed (4)

	BP	*IBM*	*LUV*	*CMG*	*ADDYY*
CapSize (millions$)	122,931	158,909	29,338	20,485	16,300
Hurst Parameter	0.505	0.496	0.629	0.567	0.552
Volatility	26.19%	21.09%	34.34%	34.07%	34.46%

10.4 Results

Instead of discussing all the stocks that were observed, we will talk about a few standout stocks here that make important points and illustrate the general trend of the 20 stocks observed. Before we discuss the stocks it is important to note one thing. In our observations, we looked at predicted profit based upon purchasing the call option and exercising it at the time of expiration and we also looked at predicted stock price vs. actual stock price. To get the predicted stock price we took the estimated cost of the option in the future and added it to the stock price on January 28th because, if the model is accurate, and in a perfect world, the price of the option would be how much the stock will go up in an arbitrage and risk free market. Naturally the two numbers will be opposite because the closer the model was at predicting the actual stock price, the less profit that call option makes and vice versa. For the purposes of our analysis here we are looking to accurately model stocks, so we will use the predicted stock price numbers.

One of the original three stocks observed, First Solar, is a solar company stock as the name notes and traded around $43 at the time of the observations. It is a low cap stock, meaning that the company is small with a total valuation of around 6 billion dollars. Small cap stocks generally have higher volatility because they are more unstable. FSLR has an estimated volatility of 56.47% and an estimated Hurst parameter of .694. Second to Amazon in the stocks we observed, FSLR had the largest difference between actual stock price and stock price predicted by the models. This was mainly due to an announcement made by China at the time of the observations that they would start purchasing more solar panels. The news caused a 20% increase in the stock price of FSLR.

As seen in the Options Pricing section, First Solar had the highest Hurst parameter of the observed stocks. In addition to being the farthest away from predicted stock price, the standard Black Scholes model was more accurate in predicting stock price than the fractional model was. Both were far off though, as seen in the summary table. First Solar's stock price graph over the past five years as seen below resembles this long-run dependence as seen in the simulated fractional Brownian motion graph with Hurst parameter of .7.

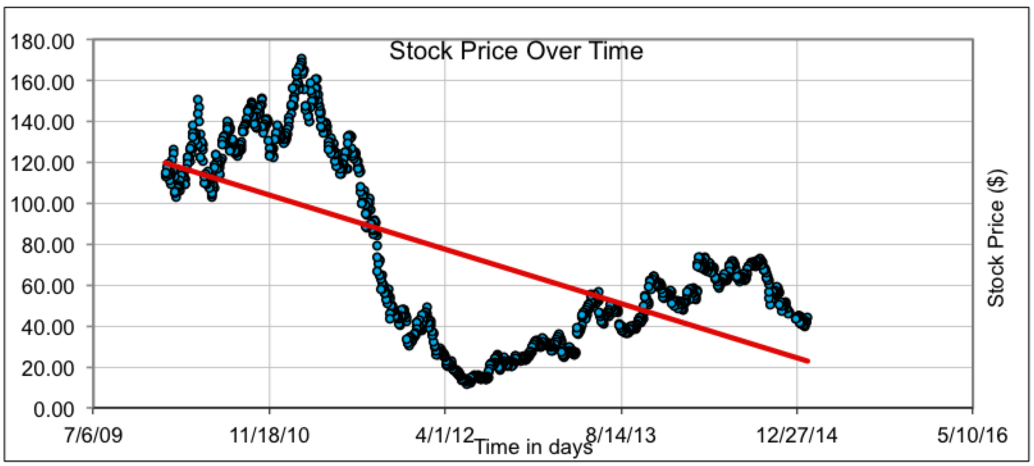

Figure 7: First Solar's stock price over time.

The next interesting stock, Accenture, is on the other far end of the spectrum. It had the lowest Hurst parameter of all the observed stocks, below .5, implying that it still remembers the past, but tends to do the opposite of what it has done. If previously Accenture went up, the parameter predicts then going forward that it will have a higher chance of decreasing. In addition Accenture had one of the lowest volatilities of the observed stocks. For Accenture, the fractional model did a more accurate job of predicting its stock price than did the standard model for both in the money and out of the money options as seen below in the table. Accenture is a notable case because as we can see from its stock price graph, it remains pretty steady and doesn't jump around much. We compare the below graph of Accenture's stock price for the past five years with the graph of simulated fractional Brownian motion for a Hurst parameter of .3 and noticed the similarities in their paths.

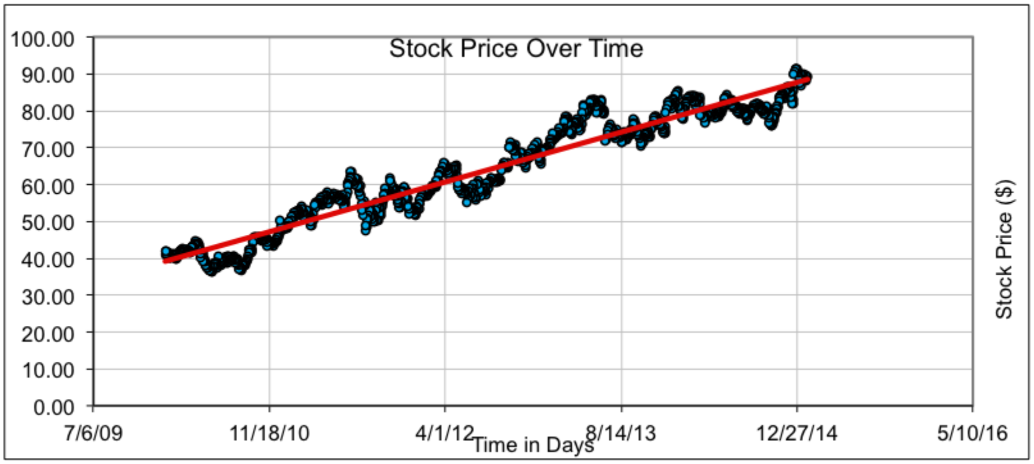

Figure 8: Accenture's stock price over time.

The third and final stock discussed here, Google, is in the middle of the spectrum as far as the Hurst parameter goes. It has an estimated H of .503, almost equaling the assumption in the original model where $H = .5$. The difference between the original model and the fractional model in predicting actual stock price is extremely small at .02%, with the original model doing slightly better. We noticed again the similarities in Google's stock price over the past five years and the simulated random walk of Brownian motion.

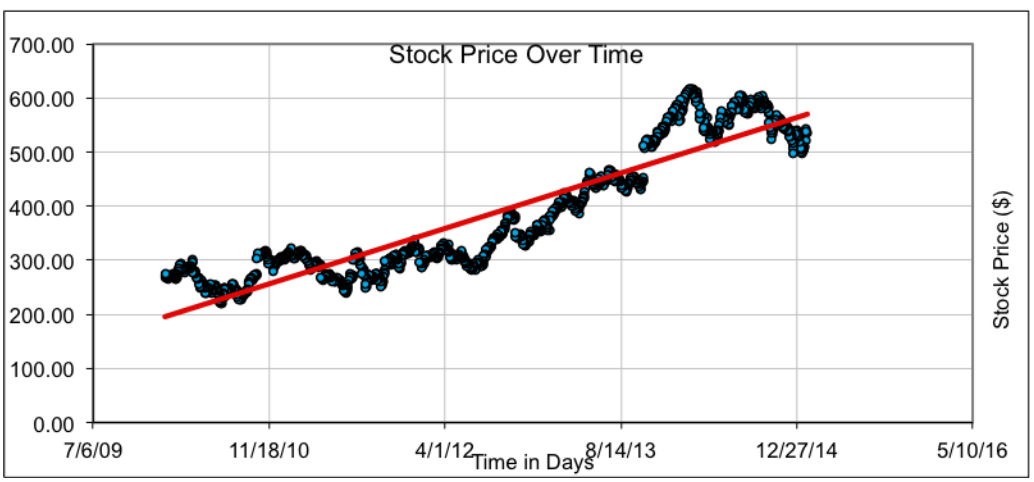

Figure 9: Google's stock price over time.

The below tables detail the average percent differences between the actual stock price and the options predicted stock price. To get the options predicted stock price we averaged the predicted option price for each method over the four weeks of data collection, and added that number to the starting stock price on January 28th, the day from which the option prices were calculated. The accompanying graph (Figure 10) plots the Hurst parameter of all 20 stocks observed against the difference in percentages from the original model and fractional model against the actual market price and shows two trendlines, one for in the money and one for out of the money options. A positive percentage indicates the fractional model was more accurate at predicting the stock price, and a negative value indicates that the original model was more accurate at predicting the stock price.

Table 5: In the Money Percentage Differences between Actual Stock Prices vs. Option Predicted Prices.

	Market Data	*Standard Bm*	*fractional Bm*
First Solar	11.58%	12.20%	13.82%
Accenture	2.27%	2.31%	1.55%
Google	N/A	5.17%	5.19%

Table 6: Out of the Money Percentage Differences between Actual Stock Prices vs. Option Predicted Prices.

	Market Data	*Standard Bm*	*fractional Bm*
First Solar	5.54%	5.62%	6.99%
Accenture	0.90%	0.88%	0.14%
Google	N/A	2.19%	2.21%

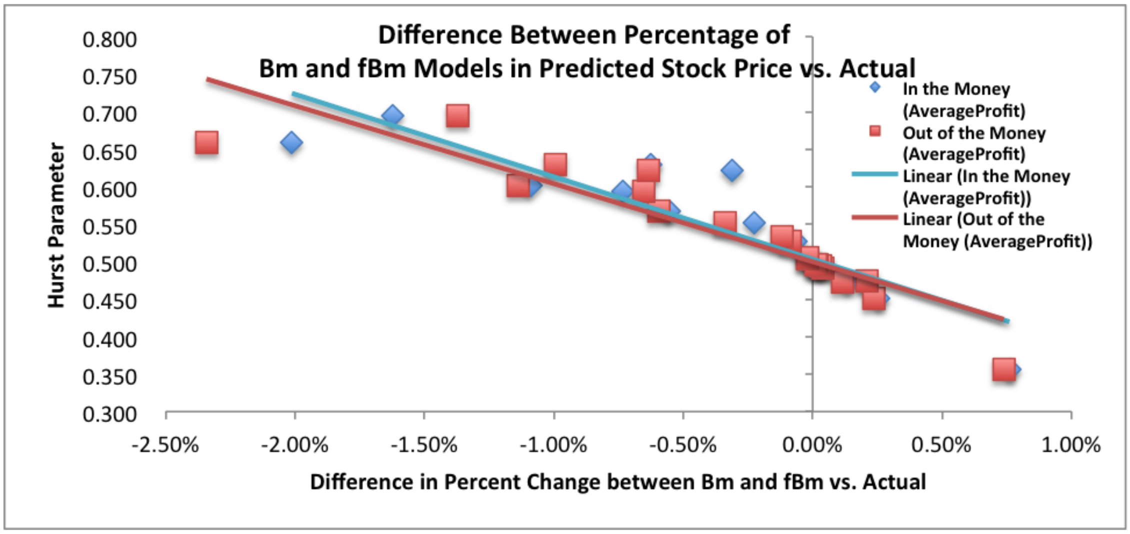

Figure 10: Hurst parameter vs. the difference in percent change in predicted stock price of Bm and fBm.

10.5 Conclusions and Explanations

In conclusion we can observe several things from the 20 stocks tracked. The first and most important question to tackle is which model is better, the original Brownian motion Black-Scholes, or the modified fractional Black-Scholes? It depends what better is defined as, but here we examined which was a more accurate predictor of stock price, and it depends on the Hurst parameter naturally. The general trend seen in the data is that for an $H < .5$, the fractional model is more accurate and becomes more accurate compared to the original model the smaller H becomes. For $H > .5$, the original model becomes a more accurate predictor of stock price, and similarly the larger H becomes, the larger the difference in the original predicted option price and the fractional predicted option price.

The natural question that arises is why is this the case? We can go back to the section on the derivative of the fractional call with respect to H and remember that in this case $\tau < 1$, therefore as H increases the call price decreases. This then predicts a lower stock price compared to the original model as H increases, but the actual stock prices don't follow this which is why the fractional model becomes worse at predicting stock prices as H increases. Economically, as H increases that means that the stock is more dependent upon the past, and therefore should be less risky because the market will know how the stock will perform going forward.

It becomes predictable. Therefore, since the stock is less risky, the predicted call price will be less than the original model because less risky investments are cheaper than risky ones because there is no uncertainty priced into the option.

An interesting outcome and conclusion of the results here is that stocks with Hurst parameters greater than .5 will give traders more profit on average. We saw that in the 20 stocks we observed that the stocks with $H > .5$ made a higher profit after executing the call option than the stocks which had a lower Hurst parameter or a value close to .5. A paper by Mitra ([7]) found a similar result by just looking at the relationship between stocks' H-value and their trading profit. Here we replicated this result, but instead of just looking at the relationships, we put a numerical value to how much profit someone should expect to make by using the fractional Black-Scholes model.

Finally it is worth noting that no matter how mathematically accurate models can become at modeling stocks and options, nothing will come close to actually predicting the market and swings as seen by First Solar's stock price jump. Furthermore just 20 stock observations out of a whole economy can't be used to create an accurate model in the difference between the original Black-Scholes and the fractional Black-Scholes, merely just to observe trends and interesting consequences.

11 References

[1] J. Robert Buchanan. *An Undergraduate Introduction to Financial Mathematics*, World Scientific Publishing Co. Pte. Ltd., 2006.

[2] Nathanael Enriquez. "A Simple Construction of the fractional Brownian motion", Stochastic Processes and their Applications 109, p. 203-223, 2004. `http://www.proba.jussieu.fr/pageperso/enriquez/PUB/SPA.pdf`.

[3] G. Grimmett, D. Stirzaker. *Probability and Random Processes*, Third Edition, Oxford University Press, 2001.

[4] Peter Gross. "Parameter Estimation for Black-Scholes Equation", Final Report, Undergraduate Research Association, 2006. `http://math.arizona.edu/~ura/061/Gross.Peter/Final.pdf`.

[5] Marcin Krzywda. "Fractional Brownian Motion and applications to financial modelling", Krakow, 2011. `http://krzywda.net/masters_2011_08_18.pdf`.

[6] Thomas Mikosch. *Elementary Stochastic Calculus with Finance in View*, Advanced Series on Statistical Science & Applied Probability, Volume 6, World Scientific Publishing Co. Pte. Ltd., 2004.

[7] S. K. Mitra, "Is Hurst Exponent Value Useful in Forecasting Financial Time Series?", Asian Social Science, Volume 8, Number 8, July 2012. `http://ccsenet.org/journal/index.php/ass/article/view/18513`.

[8] B. Qian, K. Rasheed. "Hurst Exponent and Financial Market Predictability", University of Georgia Department of Computer Science, 2004. `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.137.207&rep=rep1&type=pdf`.

[9] Karl Sigman. "IEOR 4700: Notes on Brownian Motion", Columbia University, 2006. `http://www.columbia.edu/~ks20/FE-Notes/4700-07-Notes-BM.pdf`.

A Additional Graphs Simulating the New Random Walk

A.1 The 1-Step Random Walk for Differing p-values

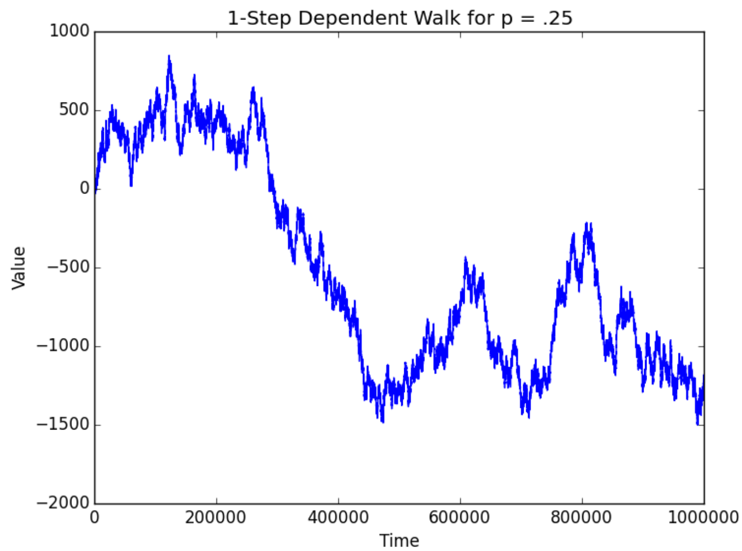

Figure 11: Simulation of 1,000,000 steps of the 1-step random walk for $p = .25$.

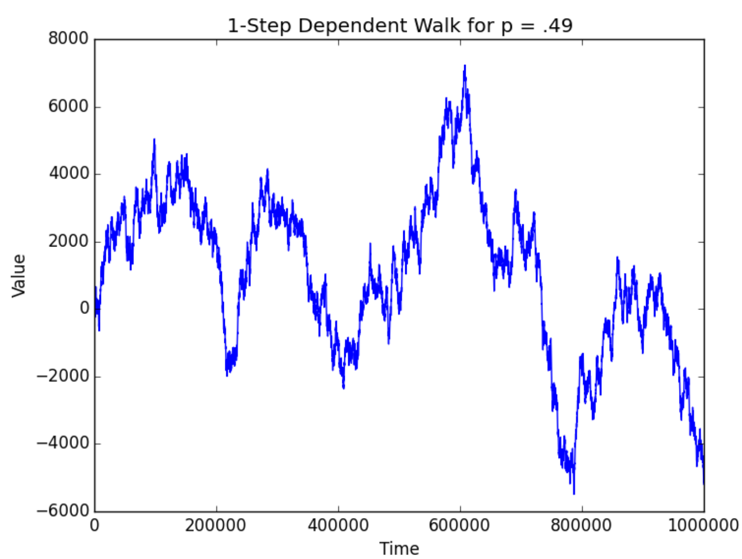

Figure 12: Simulation of 1,000,000 steps of the 1-step random walk for $p = .49$.

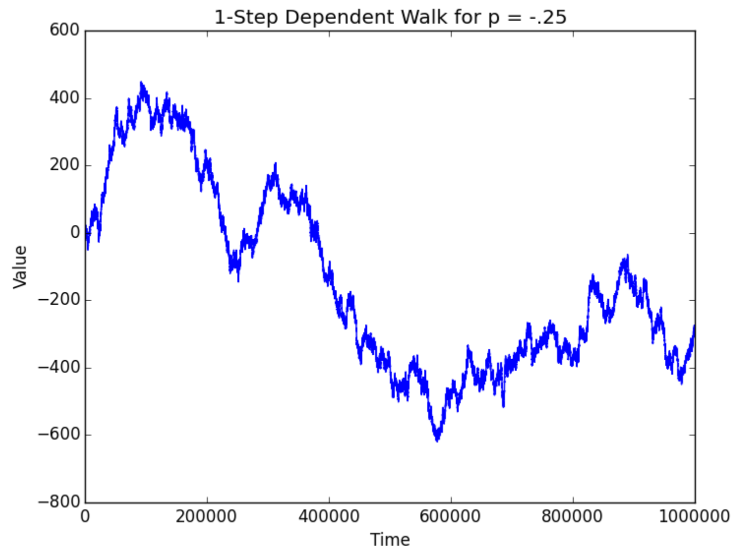

Figure 13: Simulation of 1,000,000 steps of the 1-step random walk for $p = -.25$.

Figure 14: Simulation of 1,000,000 steps of the 1-step random walk for $p = -.49$.

A.2 The 2-Step Random Walk for Differing p-values

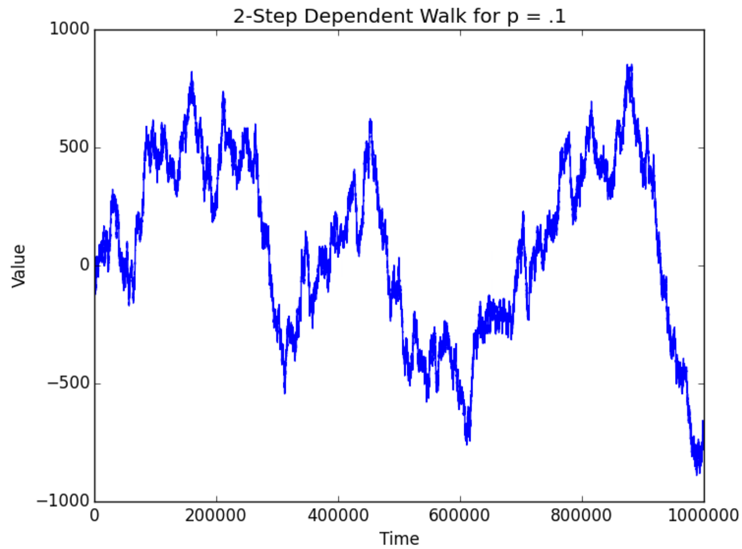

Figure 15: Simulation of 1,000,000 steps of the 2-step random walk for $p = .1$.

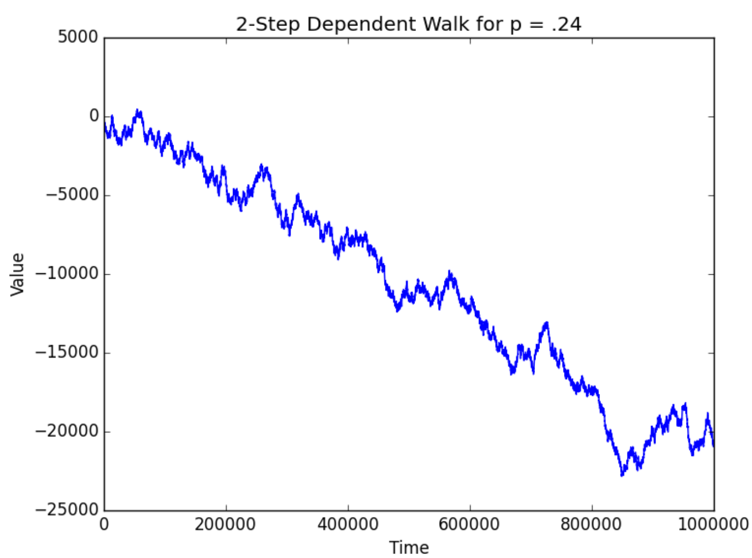

Figure 16: Simulation of 1,000,000 steps of the 2-step random walk for $p = .24$.

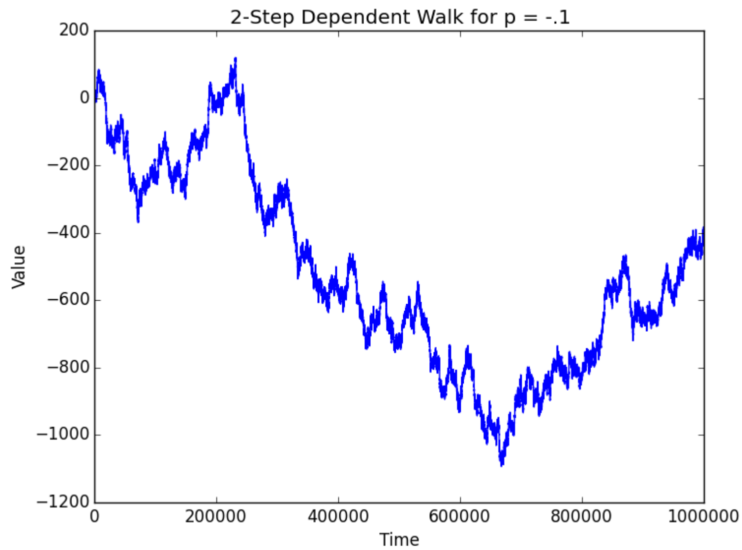

Figure 17: Simulation of 1,000,000 steps of the 2-step random walk for $p = -.1$.

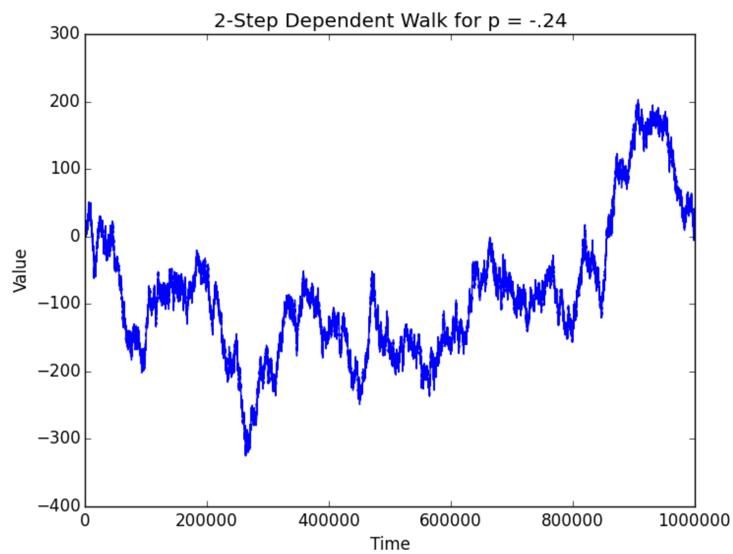

Figure 18: Simulation of 1,000,000 steps of the 2-step random walk for $p = -.24$.

A.3 The 3-Step Random Walk for Differing p-values

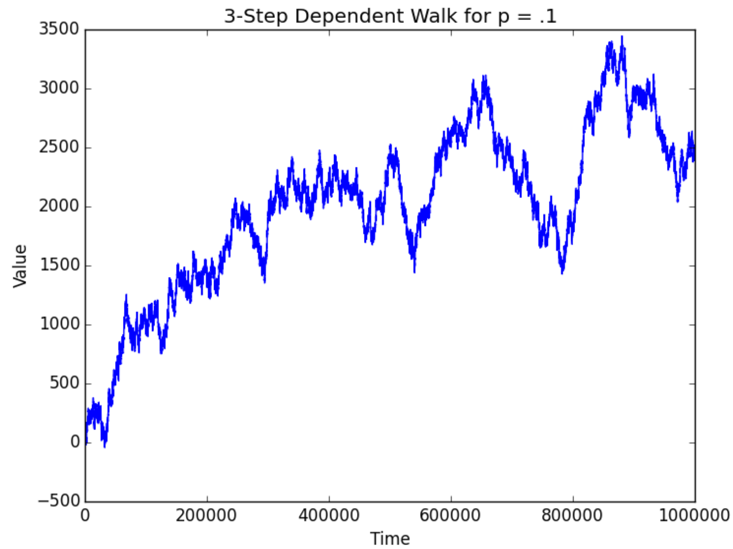

Figure 19: Simulation of 1,000,000 steps of the 3-step random walk for $p = .1$.

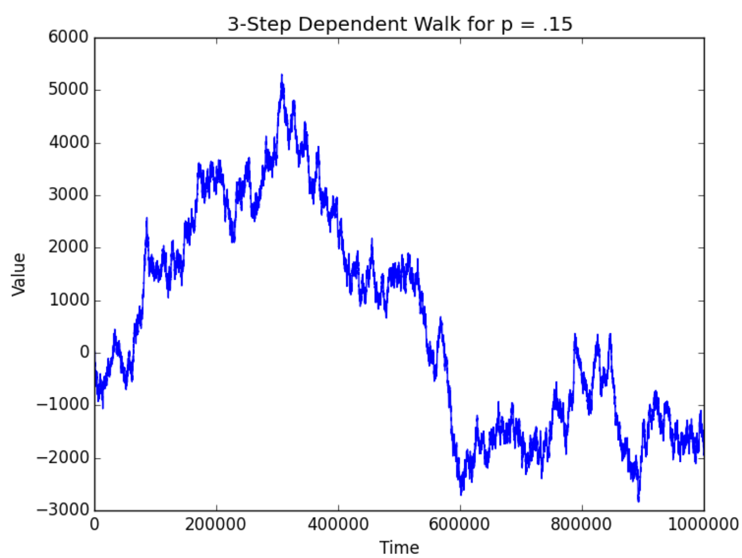

Figure 20: Simulation of 1,000,000 steps of the 3-step random walk for $p = .15$.

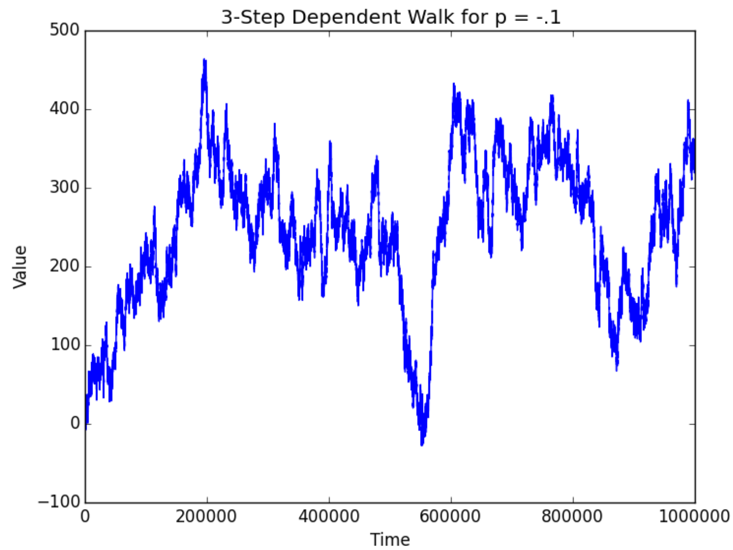

Figure 21: Simulation of 1,000,000 steps of the 3-step random walk for $p = -.1$.

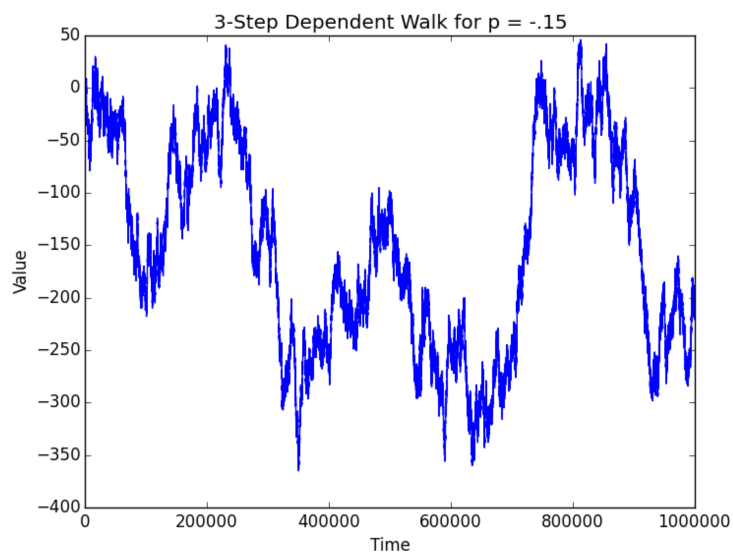

Figure 22: Simulation of 1,000,000 steps of the 3-step random walk for $p = -.15$.

B Python Models

In [44]:
```
#Code to graph and simulate the random walk, each of these is similar
#this first one is for the 1-step dependent random walk
from random import uniform
import math
import numpy as np
import matplotlib.pyplot as plt
R_n = []
R_r = []
t=1
k=1000000 #Sets the amount of steps you want to take
n=t*k
i = 0
S_n = 0
p = .49 #The value of the persistance parameter, here between -.5 and .5

while i < n: #Populates the arrays to store the values
    R_n.insert(i,0)
    R_r.insert(i,0)
    i+=1
i = 0
while i < k: #Runs for all values of our time k
    x = uniform(0,1)
    if i >0: #For all other steps this looks at the previous value
        if R_r[i-1] == 1:
            if x<(.5+p):
                R_n[i] = 1
            if x>(.5+p):
                R_n[i] = -1
        if R_r[i-1] == -1:
            if x<(.5-p):
                R_n[i] = 1
            if x>(.5-p):
                R_n[i] = -1
    if i == 0:  #Sets the first step to either be -1 or 1 with equal probability
        if x < .5:
            R_n[i] = -1
        if x>.5:
            R_n[i] = 1
    R_r[i] = R_n[i] #Stores the current value for use in the next step
    i+=1
S_n = np.cumsum(R_n) #Takes the cumulative sum of the array of -1's and 1's

plt.plot(S_n) #Graphs our cumulative sum
plt.xlabel('Time')
plt.ylabel('Value')
plt.title('1-Step Dependent Walk for p = .49')
plt.show()
```

In [42]:
```
#Does the same as above, here it is a 2-Step random walk
from random import uniform
import math
import numpy as np
import matplotlib.pyplot as plt
R_n = []
```

```
R_r = []
t=1
k=1000000
n=t*k
i = 0
S_n = 0
r = -.49

while i < n:
    R_n.insert(i,0)
    R_r.insert(i,0)
    i+=1
i = 0
while i < k:
    x = uniform(0,1)
    if i >1:
        if R_r[i-2] == 1:
            if R_r[i-1] ==1:
                if x<(.5+2*r):
                    R_n[i] = 1
                if x>(.5+2*r):
                    R_n[i] = -1
            if R_r[i-1] == -1:
                if x < .5:
                    R_n[i] = -1
                if x>.5:
                    R_n[i] = 1
        if R_r[i-2] == -1:
            if R_r[i-1] ==-1:
                if x<(.5-2*r):
                    R_n[i] = 1
                if x>(.5-2*r):
                    R_n[i] = -1
            if R_r[i-1] == 1:
                if x < .5:
                    R_n[i] = -1
                if x>.5:
                    R_n[i] = 1
    if i == 1:
        if x < .5:
            R_n[i] = -1
        if x>.5:
            R_n[i] = 1
    if i == 0:
        if x < .5:
            R_n[i] = -1
        if x>.5:
            R_n[i] = 1
    R_r[i] = R_n[i]
    i+=1
S_n = np.cumsum(R_n)

plt.plot(S_n)
plt.xlabel('Time')
```

```
    plt.ylabel('Value')
    plt.title('2-Step Dependent Walk for p = -.49')
    plt.show()
```

In [9]:
```
#A 3-Step random walk
from random import uniform
import math
import numpy as np
import matplotlib.pyplot as plt
R_n = []
R_r = []
t=1
k=1000000
n=t*k
i = 0
S_n = 0
r = .15

while i < n:
    R_n.insert(i,0)
    R_r.insert(i,0)
    i+=1
i = 0
while i < k:
    x = uniform(0,1)
    if i >2:
        if R_r[i-3] == 1:
            if R_r[i-2] ==1:
                if R_r[i-1] ==1:
                    if x<(.5+3*r):
                        R_n[i] = 1
                    if x>(.5+3*r):
                        R_n[i] = -1
                if R_r[i-1] == -1:
                    if x<(.5+r):
                        R_n[i] = 1
                    if x>(.5+r):
                        R_n[i] = -1
            if R_r[i-2] == -1:
                if R_r[i-1] ==1:
                    if x<(.5+r):
                        R_n[i] = 1
                    if x>(.5+r):
                        R_n[i] = -1
                if R_r[i-1] == -1:
                    if x<(.5-r):
                        R_n[i] = 1
                    if x>(.5-r):
                        R_n[i] = -1
        if R_r[i-3] == -1:
            if R_r[i-2] ==1:
                if R_r[i-1] ==1:
                    if x<(.5+r):
                        R_n[i] = 1
```

```
                        if x>(.5+r):
                            R_n[i] = -1
                    if R_r[i-1] == -1:
                        if x<(.5-r):
                            R_n[i] = 1
                        if x>(.5-r):
                            R_n[i] = -1
                if R_r[i-2] == -1:
                    if R_r[i-1] ==1:
                        if x<(.5-r):
                            R_n[i] = 1
                        if x>(.5-r):
                            R_n[i] = -1
                    if R_r[i-1] == -1:
                        if x<(.5-3*r):
                            R_n[i] = 1
                        if x>(.5-3*r):
                            R_n[i] = -1
        if i < 3:
            if x < .5:
                R_n[i] = -1
            if x>.5:
                R_n[i] = 1
        R_r[i] = R_n[i]
        i+=1
    S_n = np.cumsum(R_n)

    plt.plot(S_n)
    plt.xlabel('Time')
    plt.ylabel('Value')
    plt.title('3-Step Dependent Walk for p = .15')
    plt.show()
```

```
In [ ]:
```