ON THE IRRATIONALITY OF A DIVISOR FUNCTION SERIES

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Abstract

Here, we show, unconditionally for $k = 3$, and on the prime $k$-tuples conjecture for $k \geq 4$, that $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n}$ is irrational, where $\sigma_k(n)$ denotes the sum of the $k$th powers of the divisors of $n$.

1. Introduction

For a positive integer $n$ put

$$\sigma_k(n) = \sum_{d|n} d^k$$

for the sum of the $k$th powers of the (positive) divisors of $n$. In [4], Erdős and Kac showed that the series

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational for both $k = 1$ and $k = 2$. The problem is mentioned also in [5], wherein it is stated that the method does not seem to extend to $k \geq 3$, and it appears as B14 in [6]. Let

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us quickly give proofs of the irrationality of these series when \( k = 0, 1, 2 \). Assume that the given series is \( A/B \), where \( A \) and \( B \) are positive integers. Multiplying by \((n - 1)!\), where \( n > B \), we get that

\[
\sum_{j=1}^{n-1} \sigma_k(j) \frac{(n - 1)!}{j!} + \frac{\sigma_k(n)}{n} + \frac{\sigma_k(n + 1)}{n(n + 1)} + \sum_{j=2}^{n - 1} \frac{\sigma_k(n + j)}{n(n + 1) \cdots (n + j)} = A(n - 1)! B^{-1}.
\]

The first of the above sums is an integer and the right hand side is also an integer. The second sum is positive and, for \( k \leq 2 \), its size is

\[
\leq \frac{\sigma_2(n + 2)}{n(n + 1)(n + 2)} + \sum_{j=3}^{n-1} \frac{\sigma_2(n + j)}{n(n + 1) \cdots (n + j)} \ll \frac{1}{n} + \sum_{m \geq n} \frac{1}{m^2} \ll \frac{1}{n},
\]

so that this term belongs to the interval \([0, c/n]\), where \( c \) is an absolute constant. We shall take \( n \) to be a prime \( n \equiv 1 \pmod{4} \) such that \((n + 1)/2\) has no prime factor \(< y\), where \( y \) is a large positive real number. Such primes exist by Dirichlet’s theorem on primes in arithmetical progressions. Then

\[
\frac{\sigma_1(n + 1)}{n(n + 1)} \leq \frac{c_1 \log \log n}{n}
\]

for some constant \( c_1 \), and

\[
\frac{5}{4} \leq \frac{\sigma_2(n + 1)}{n(n + 1)} \leq \frac{5(n + 1)}{4n} \prod_{q \geq y} \left( 1 + \frac{1}{q^2} + \cdots \right)
\]

\[
\leq \frac{5(n + 1)}{4n} \exp \left( \sum_{q \geq y} \frac{1}{q(q - 1)} \right) = \frac{5(n + 1)}{4n} e^{O(1/y)}
\]

\[
= \frac{5}{4} + O \left( \frac{1}{y} \right)
\]

whenever \( n > y \). Thus, when \( k = 1 \), we get that \( \sigma_1(n)/n = 1 + 1/n \), and we conclude that there is an integer in the interval \([1/n, (c + 1 + c_1 \log \log n)/n]\), while when \( k = 2 \) we get that \( \sigma_2(n)/n = n + 1/n \), and so there exists an integer in the interval \([1/4, 1/4 + c/n + c_2/y]\), where \( c_2 \) is the constant implied by the \( O \) symbol in (1). Choosing \( n \) (and \( y \)) to be sufficiently large, we get a contradiction in the case \( k = 1 \) (and \( k = 2 \)), which completes the proof.

In this paper, we make a modest contribution to the problem, proving two results about the irrationality of the above series for \( k \geq 3 \).

**Theorem 1.** The sum of the series

\[
\sum_{n=1}^{\infty} \frac{\sigma_3(n)}{n!}
\]

is irrational.
We recall the statement of the Prime $k$-tuples Conjecture (see [3, 7, 9]), which is due to Dickson.

**Conjecture 1.** For any $k \geq 2$, let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be integers with $a_i > 0$ for each $i = 1, \ldots, k$. Suppose that for every prime number $p$ there exists an integer $n$ such that $\prod_{i=1}^{k} (a_in + b_i)$ is not a multiple of $p$. Then there exist infinitely many positive integers $n$ such that $p_i = a_in + b_i$ is prime for all $i = 1, \ldots, k$.

We have the following result for any positive integer $k$.

**Theorem 2.** The Prime $k$-tuples Conjecture implies that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational.

Throughout this paper, we use the Landau symbols $O$ and $o$ as well as Vinogradov’s symbols $\gg, \ll$ and $\sim$ with their regular meaning. The constants implied by these symbols depend at most on $k$. We use $p$ and $q$ to denote prime numbers.

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## 2. Proof of Theorem 1

Our main tool is the following well-known theorem of Chen (see [1], [2]).

**Theorem 3.** Let $a$ be an integer. There exists $x_a$ such that for $x > x_a$, the interval $[x/2, x]$ contains $\gg x_a/\varphi(a)^2(\log x)^2$ prime numbers $p \equiv 1 \pmod{a}$ such that $q = (p + 1)/2$ is either a prime or a product of two primes each one of which exceeds $x^{1/10}$.

A proof containing the basic ideas for this theorem appears, for example, in Chapter 11 of [8]. Chen actually proved that for large even integers $N$ there are $\gg N/(\log N)^2$ primes $p$ such that $N - p$ is either a prime or a product of two large primes. Easy and well-known modifications of Chen’s argument yield the above theorem.

To prove our Theorem 1 we assume that the sum of the series shown at (2) is rational and deduce a contradiction. We write it, as we did for $k = 0$, 1 and 2 as $A/B$, multiply
across by \((n - 1)!\) for some large \(n\) (with \(n > B\)), and obtain
\[
\frac{\sigma_3(n)}{n} + \frac{\sigma_3(n + 1)}{n(n + 1)} + \frac{\sigma_3(n + 2)}{n(n + 1)(n + 2)} + \sum_{j \geq 3} \frac{\sigma_3(n + j)}{n(n + 1) \cdots (n + k)} = (n - 1)! \left( \frac{A}{B} - \sum_{j = 1}^{n-1} \frac{\sigma_k(j)}{j!} \right),
\]
where the right hand side is an integer. In what follows, we shall exploit the above relation. Since \(\sigma_3(n) \ll n^3\), we have
\[
\sum_{j \geq 4} \frac{\sigma_3(n + j)}{n(n + 1) \cdots (n + j)} \ll \frac{1}{n} + \sum_{m \geq n} \frac{1}{m^2} \ll \frac{1}{n}.
\]
Furthermore, the sum appearing on the left hand side of formula (3) above is a positive integer. We now let \(y\) be a large positive real number which we shall fix later, and take \(a = 72 \prod_{5 \leq p \leq y} p\). Note that, by Chebyshev’s bounds, this means that \(y \asymp \log a\). We let \(x\) be large compared to \(a\) and \(m \in [x/2a, x/a]\) be such that \(p = am + 1\) is prime and \(q = (p + 1)/2 = (am + 2)/2\) is either a prime or a product of two primes \(q_1q_2\) each exceeding \(x^{1/10}\). Choose \(n\) to be the prime \(n = p\). We then note that
\[
\frac{\sigma_3(n)}{n} = n^2 + \frac{1}{n}.
\]
Furthermore, writing \(n + 2 = am + 3 = 3t\), we have that \(t\) is coprime to all primes \(q \leq y\), and so obtain the inequalities
\[
\frac{28}{27} < \frac{\sigma_3(n + 2)}{n(n + 1)(n + 2)} = \frac{28}{27} \frac{\sigma_3(t)}{t^3} \left( 1 + O \left( \frac{1}{n} \right) \right)
< \frac{28}{27} \prod_{p > y} \left( 1 - \frac{1}{p^3} \right)^{-1} \left( 1 + O \left( \frac{1}{n} \right) \right)
< 1 + \frac{1}{27} + O \left( \frac{1}{y^2} \right),
\]
since \(y < \sqrt{x}\). Combining this with our estimate for the tail of the series we have, for \(n\) a prime as above,
\[
\frac{\sigma_3(n)}{n} + \frac{\sigma_3(n + 2)}{n(n + 1)(n + 2)} + \sum_{j \geq 3} \frac{\sigma_3(n + j)}{n(n + 1) \cdots (n + j)} = A_n + \frac{1}{27} + O \left( \frac{1}{y^2} \right),
\]
where \(A_n\) is a positive integer.

We now consider the remaining term \(\sigma_3(n + 1)/(n(n + 1))\).

Assume first that for some large \(x\) there is a prime \(p = am + 1 \in [x/2, x]\) such that \(q = (p + 1)/2 = (a/2)m + 1\) is prime. Then
\[
\frac{\sigma_3(n + 1)}{n(n + 1)} = \frac{\sigma_3(2q)}{2q(2q - 1)} = \frac{9(q^2 + 1)}{2q(2q - 1)}
= \frac{9}{4q} + \frac{9}{8q} + \frac{9q - 4}{8q(2q - 1)} = B_n + \frac{3}{8} + O \left( \frac{1}{n} \right).
\]
with some integer $B_n$, where we have used the fact that $q = (a/2)m + 1 \equiv 1 \pmod{4}$, because $8 \mid a$. Summing up everything, we find that
\[
\frac{1}{27} + \frac{3}{8} + O\left(\frac{1}{y^2}\right) \in \mathbb{Z},
\]
which is impossible if $y$ is chosen to be sufficiently large.

So, we are left with the more interesting part of the problem where $q = q_1q_2$, where $q_1 > q_2 > x^{1/10}$ holds for all the $\gg xa/\varphi(a)^2(\log x)^2$ choices of primes $p = am + 1 \in [x/2, x]$ guaranteed by Chen’s Theorem 3.

We put
\[
M = \left[\frac{\sigma_3(n + 1)}{n(n + 1)}\right] = \left[\frac{9\sigma_3(q)}{2q(2q - 1)}\right].
\]
Note that since $q_2 \leq q^{1/2}$, we have that
\[
\frac{\sigma_3(q)}{2q(2q - 1)} = \frac{q^3 + q^3 + q^3 + 1}{2q(2q - 1)} = \frac{q^3 + q^3 + O(q^{3/2})}{2q(2q - 1)} = \frac{q^3 + q^3}{2q(2q - 1)} + O\left(\frac{1}{q^{1/2}}\right).
\]
Thus, using also the previous calculations of fractional parts (see (4)), we find that for large $x$,
\[
M + 26 = \frac{27}{2q} - \frac{c_0}{y^2} \leq \frac{9q^3 + 9q^3}{2q(2q - 1)} \leq M + 26 + c_0 \frac{y^2}{y^2}
\]
holds for all large $x$ with some constant $c_0 > 0$, an inequality which can be rewritten as
\[
q^2 \left(4M + 3 - 9q + \frac{23}{27} - \frac{2q(m + O(1))}{q^2} - \frac{4c_0}{y^2}\right) \leq 9q^3
\]
\[
\leq q^2 \left(4M + 3 - 9q + \frac{23}{27} - \frac{2q(m + O(1))}{q^2} + \frac{4c_0}{y^2}\right).
\]
From this inequality and the fact that $q_1 < qx^{-1/10}$, we find that
\[
\frac{M}{q} = \frac{9(q^3 + q^3)}{2q^2(2q - 1)} + O\left(\frac{1}{q^4}\right) = \frac{9}{4} + O\left(\frac{1}{x^{3/10}}\right)
\]
so
\[
-\frac{2q(M + O(1))}{q^2} = -\frac{9}{2} + O\left(\frac{1}{x^{3/10}}\right).
\]
Hence, we deduce that
\[
4M + 3 - 9q + \frac{23}{27} - \frac{2q(M + O(1))}{q^2} = L + \frac{19}{54} + O\left(\frac{1}{x^{3/10}}\right)
\]
holds with some non-negative integer $L$. Thus, for large $x$, provided that $y < x^{3/20}$, we get that
\[
\frac{q^2}{9} \left(L + \frac{19}{54} - \frac{5c_0}{y^2}\right) < q_1^3 < \frac{q^2}{9} \left(L + \frac{19}{54} + \frac{5c_0}{y^2}\right),
\]
which is equivalent to
\[
\frac{q_2^2}{9} \left( L + \frac{19}{54} - \frac{5c_0}{y^2} \right) < q_1 < \frac{q_2^2}{9} \left( L + \frac{19}{54} + \frac{5c_0}{y^2} \right).
\] (5)

The above inequalities certainly tell us that \( L \geq 0 \), provided that \( y \) is sufficiently large. Further, the left hand side of the above inequality is \( > \frac{q_2^2}{27} \) if \( y^2 > 220c_0 \), so if \( y \) satisfies the above inequality and \( x \) is large, then \( \frac{q_2^2}{27} \leq q_1 \leq x/q_2 \), and therefore \( q_2 \leq 3x^{1/3} \). Now fix \( q_2 \). Then the left hand side of the above inequality is \( \geq (L + 1/3)q_2^2/27 \) and the middle term satisfies \( q_1 \leq x/q_2 \). Thus, \( L + 1/3 \leq 27x/q_2^2 \). Since \( L \geq 0 \) is an integer, it follows that the number of possibilities for \( L \) is
\[
1 + \left\lceil \frac{27x}{q_2^2} \right\rceil \leq \frac{81x}{q_2^2}.
\] (6)

We now fix also \( L \geq 0 \). Then \( q_1 \) is a prime in the interval shown at (5) above such that \( q_1q_2 \equiv 1 \pmod{a} \) and \( 2q_1q_2 - 1 = p \) is a prime. By the Brun sieve, the number of such primes is
\[
\ll \frac{c_0q_2^2}{y^2} \left( \frac{a}{\varphi(a)^2} \right) \cdot \frac{1}{(\log(10c_0q_2^2/(9y^2a)))^2}.
\]

Since \( q_2 > x^{1/10} \), it follows that for large \( x \) we have \( 10c_0q_2^2/(9y^2a) > x^{1/6} \). Thus, the number of possibilities for \( q_1 \) once \( q_2 \) and \( L \) are fixed is
\[
\ll \frac{1}{y^2} \cdot \frac{a}{\varphi(a)^2} \cdot \frac{x}{(\log x)^2} \cdot \frac{q_2^2}{q_2}.
\]

Summing up over all the possibilities for \( L \) shown at (6), we get that the total number of possibilities for \( q_1 \) when \( q_2 \) is fixed is
\[
\ll \frac{1}{y^2} \cdot \frac{a}{\varphi(a)^2} \cdot \frac{x}{(\log x)^2} \cdot \frac{1}{q_2},
\]
and now summing up the above bound over all primes \( q_2 \in [x^{1/10}, 3x^{1/3}] \), we get that the total number of possibilities for \( p \) is
\[
\ll \frac{1}{y^2} \cdot \frac{a}{\varphi(a)^2} \cdot \frac{x}{(\log x)^2} \sum_{x^{1/10} \leq q_2 \leq 3x^{1/3}} \frac{1}{q_2} \ll \frac{1}{y^2} \cdot \frac{a}{\varphi(a)^2} \cdot \frac{x}{(\log x)^2} \cdot \frac{c_2 x}{q_2},
\] (7)

where the fact that the last sum above is \( O(1) \) follows from Mertens’s estimate for the summatory function of the reciprocal of the primes. Let \( c_1 \) be the constant implied in the Vinogradov symbol from Chen’s Theorem 3 and \( c_2 \) be the constant implied in the last Vinogradov symbol in (7). Comparing the estimates for the number of primes \( p \) under scrutiny we get
\[
\frac{c_1ax}{\varphi(a)^2(\log x)^2} \leq \frac{c_2ax}{\varphi(a)^2y^2(\log x)^2}
\]
leading to \( y^2 < c_3 \), where \( c_3 = c_2/c_1 \). Choosing \( y \) larger than this we complete the proof of Theorem 1.
3. Proof of Theorem 2

Let \( k \geq 4 \). For \( i = 1, \ldots, k \) we let \( Q_i(X), R_i(X) \in \mathbb{Z}[X] \) be the polynomials given by the division algorithm:

\[
(X + i)^k = Q_i(X)(X + 1) \cdots (X + i) + R_i(X) \quad i = 1, \ldots, k,
\]

where \( \deg Q_i(X) = k - i \) and \( \deg R_i(X) \leq i - 1 \). Note that when \( i = k \) we have that \( Q_k(X) = 1 \). For each of those \( i = 1, \ldots, k \) for which \( Q_i(-i) \neq 0 \) choose distinct primes \( p_i > k \) such that \( p_i \nmid \sigma_k(i)Q_i(-i) \). (For any other \( i \), for notational purposes, take \( p_i = 1 \).)

As we have seen, \( Q_k(-k) = 1 \neq 0 \), so that \( p_k \nmid \sigma_k(k)Q_k(-k) \) can be arranged simply by choosing \( p_k > \sigma_k(k) \).

Let \( P = \prod_{i=1}^{k} p_i \) and let \( m \) be a positive integer such that

\[
\frac{(k!)^2}{i}m + 1 \equiv p_i \pmod{p_i^2} \quad i = 1, \ldots, k.
\]

By the Chinese Remainder Theorem, there are infinitely many such positive integers \( m \) and they form the arithmetic progression \( m_0 \pmod{P^2} \), where \( m_0 \) is the first positive integer in this progression. Given such an \( m \), we choose \( n = (k!)^2m \) and write \( m = m_0 + \ell P^2 \) with some nonnegative integer \( \ell \). Then

\[
n + i = i \left( \frac{(k!)^2}{i}m + 1 \right) = ip_i \left( \frac{(k!)^2 P^2}{i} \frac{\ell}{p_i} + \frac{(k!)^2 m_0}{i} + 1 \frac{1}{p_i} \right).
\]

Put

\[
A_i = \frac{(k!)^2 P^2}{i} \quad \text{and} \quad B_i = \frac{(k!)^2 m_0}{i} + 1 \frac{1}{p_i}
\]

for \( i = 1, \ldots, k \).

One checks easily that we can apply the Prime \( k \)-tuples Conjecture 1 to the linear polynomials \( A_i\ell + B_i \). Indeed, the prime numbers dividing \( A_i \) are exactly the primes \( p \leq k \) together with the primes \( p_i \). Since \( B_i \mid (k!)^2 m_0/i + 1 \), we see that \( B_i \) is coprime to all primes \( p \leq k \). Further, \( B_i \equiv 1 \pmod{p_i} \) from the way \( m_0 \) was chosen. To see that \( B_i \) is coprime to \( p_j \) if \( j \neq i \) assume otherwise. Then \( p_j \mid ((k!)^2/j)m_0 + 1 \) and \( p_j \mid B_i \mid ((k!)^2/i)m_0 + 1 \). Hence, \( p_j \mid (k!)^2 m_0 / i + j \) and \( p_j \mid (k!)^2 m_0 + i \), so \( p_j \mid (j - i) \), but this is false because \( p_j > k \). Thus, the conditions for the applicability of the Prime \( k \)-tuples Conjecture 1 are fulfilled and we can let \( \ell \) be large and such that \( A_i\ell + B_i = q_i \) are all primes for \( i = 1, \ldots, k \). Put \( a_i = ip_i \) and note that if \( m = m_0 + \ell P^2 \), then \( n + i = a_i q_i \) for \( i = 1, \ldots, k \).

Now assume that the series

\[
\sum_{j=1}^{\infty} \frac{\sigma_k(j)}{j!} = A \frac{B}{B}
\]
with $A$ and $B$ coprime integers. Suppose $n$ is sufficiently large, and in particular $n > B$. Multiplying across by $n!$, we get

\[
\sum_{j \leq n} \frac{\sigma_k(j)}{j!} \cdot \frac{n!}{j!} + \sum_{i=1}^{k} \frac{\sigma_k(n+i)}{(n+1) \cdots (n+i)} + \sum_{j \geq k+1} \frac{\sigma_k(n+j)}{(n+1) \cdots (n+j)} \in \mathbb{Z}. \tag{8}
\]

The first sum above is an integer, while the last sum is positive and, since $\sigma_k(n) \ll n^k$,

\[
\sum_{j \geq k+1} \frac{\sigma_k(n+j)}{(n+1) \cdots (n+j)} = \frac{\sigma_k(n+k+1)}{(n+1) \cdots (n+k+1)} + \sum_{j \geq k+2} \frac{\sigma_k(n+j)}{(n+1) \cdots (n+j)} < \frac{1}{n} + \sum_{j \geq k+2} \frac{1}{(n+j)^2} \ll \frac{1}{n}.
\]

As for the intermediate terms, since $q_i$ is prime we have

\[
\frac{\sigma_k(n+i)}{(n+1) \cdots (n+i)} = \frac{\sigma_k(a_i)(q_i^k + 1)}{(n+1) \cdots (n+i)} = \frac{\sigma_k(a_i)}{a_i^k} \left( \frac{(n+i)^k + a_i^k}{(n+1) \cdots (n+i)} \right) = \frac{\sigma_k(a_i)}{a_i^k} Q_i(n) + \frac{R_i(n) + a_i^k}{(n+1) \cdots (n+i)} = \frac{\sigma_k(a_i)}{a_i^k} Q_i(n) + O \left( \frac{1}{n} \right),
\]

where for the above error term we used the fact that $\deg R_i \leq i - 1$. Since $n \equiv -i \pmod{p_i}$, we get that $Q_i(n) \equiv Q_i(-i) \pmod{p_i}$ so that $Q_i(n) = Q_i(-i) + p_i \ell_i(n)$, for some integers $\ell_i(n)$. Thus,

\[
\frac{\sigma_k(n+i)}{(n+1) \cdots (n+i)} = \frac{\sigma_k(ip_i)}{(ip_i)^k} (Q_i(-i) + \ell_i(n)p_i) + O \left( \frac{1}{n} \right).
\]

We add everything together, multiply formula (8) by $(k!)^k P_i^{k-1}$ and get

\[
\sum_{i=1}^{k} P_i^{k-1} \sigma_k(ip_i) \frac{(k!)^k}{i^k} \frac{1}{p_i} (Q_i(-i) + p_i \ell_i(n)) + O \left( \frac{1}{n} \right) \in \mathbb{Z},
\]

where

\[
P_i = P_p^{-1} = \prod_{1 \leq j \leq k, j \neq i} p_j.
\]

From this it follows that

\[
\sum_{i=1}^{k} P_i^{k-1} \frac{(k!)^k}{i^k} \frac{\sigma_k(ip_i)Q_i(-i)}{p_i} + O \left( \frac{1}{n} \right) \in \mathbb{Z}.
\]
At this moment, we observe that the sum on the left does not depend on \( n \) and so
\[
\sum_{1 \leq i \leq k} P_i^{k-1} \left( \frac{(k!)^k \sigma_k(i p_i)}{i^k} \right) Q_i(-i) \in \mathbb{Z}.
\]
Using also the fact that \( \sigma_k(i p_i) = \sigma_k(i) \sigma_k(p_i) = \sigma_k(i)(p_i^k + 1) \), we get that the above relation implies
\[
\sum_{1 \leq i \leq k} P_i^{k-1} \left( \frac{(k!)^k \sigma_k(i)}{i^k} \right) Q_i(-i) \in \mathbb{Z}.
\]
This is a non-empty (since \( Q_k(-k) = 1 \)) sum of non-zero rational numbers whose reduced denominators are distinct primes. Thus, the above sum cannot be an integer. The proof of Theorem 2 is complete.

**Note Added: March 2007.** Shortly after this paper was submitted, we learned about the recent appearance of the paper [10], where the same two results as the present ones have been obtained. The proof of the case \( k = 3 \) in [10] also uses sieve methods but is rather different, whereas the conditional proof for larger \( k \)'s is similar to ours. We thank Professor Igor Shparlinski for pointing out this reference to us.

**References**


