Sturm-Liouville Oscillation Theory for Differential Equations and Applications to Functional Analysis

by

ZHAONING WANG

Mihai Stoiciu, Advisor

A thesis submitted in partial fulfillment
of the requirements for the
Degree of Bachelor of Arts with Honors
in Mathematics

WILLIAMS COLLEGE
Williamstown, Massachusetts USA

May 11, 2011

Abstract

We study the connection between second-order differential equations and their corresponding difference equations. With this connection in mind, we investigate quantitative and qualitative properties of the zeros of the solutions of differential/difference equations and of the eigenvalues of the associated Jacobi matrices. In particular, we study various applications of the Sturm-Liouville Oscillation Theory to differential equations and spectral theory.

Zhaoning (Nancy) Wang
Williams College
Williamstown, MA 01267

Acknowledgements

For months of guidance, inspiration, and insight, I am most grateful to my advisor, Prof. Mihai Stoiciu for his excellent advising and confidence in me. I am also very appreciative of the helpful feedback I received from my second reader, Prof. Stewart D. Johnson. I would also like to thank the professors and students of the Williams College Mathematics Department, who have provided me with the challenges and the support that have made my four years at Williams meaningful and rewarding. Finally, I would like to thank my parents for their unconditional love and support, which have made it possible for me to accomplishment my goals.

Contents

1 Introduction **5**

2 First Order Systems **7**

2.1 Existence and Uniqueness of Solutions . . . 7

2.2 Properties of the First Order Systems . . . 12

3 Second Order Scalar Equations **19**

3.1 Existence and Uniqueness of Solutions . . . 19

3.1.1 Examples . . . 23

3.2 Differentiable Dependence . . . 29

3.3 The Prüfer Transformations . . . 47

3.3.1 Definition . . . 47

3.3.2 Examples . . . 49

3.4 Oscillation . . . 49

3.4.1 Definitions . . . 50

3.4.2 Oscillation Criteria . . . 50

3.5 Sturm Comparison Theorem . . . 54

4 Sturm Oscillation and Comparison Theorems **55**

4.1 Basics . . . 55

4.1.1 Construction of Difference Equations from a Jacobi Matrix . . . 55

4.2 Sturm Oscillation and Comparison Theorems . . . 57

5 Connection between the Two Approaches **57**

5.1 Construction of the Jacobi Matrix from Differential Equations . . . 57

5.2 Examples . . . 60

6 References **65**

Appendices **66**

A. Calculations for the Prüfer Transformation Example . . . 66

B. Calculations for the Difference/Differential Equations . . . 67

C. Calculations for the Difference/Differential Equations . . . 68

1 Introduction

In 1836 and 1837, Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882) published a series of papers on second order linear ordinary differential equations of the form

$$-(py')' + qy = f \text{ on } J \tag{1.1}$$

together with the initial conditions

$$y(c) = h, \quad (py')(c) = k, \quad c \in J, \quad h, k \in \mathbb{C} \tag{1.2}$$

where

$$J = (a, b), \quad -\infty \leq a < b \leq +\infty, \quad \frac{1}{p},\ q,\ f : J \to \mathbb{C}. \tag{1.3}$$

This marked the beginning of the investigation of the "Sturm-Liouville Problems" and the establishment of the Sturm-Liouville Theory.

The discrete analog of the differential equation above is the difference equation defined by the following recurrence relations

$$(hu)_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = xu \tag{1.4}$$

for $n = 1, 2, \cdots$ with $u_0 = 0$, where h is some operator on u, x is a parameter, u, u_n are functions of x for every n.

Among all the Ordinary Differential Equations (ODE), there are some equations that are explicitly solvable, meaning that for those equations, we are able to find a closed-form expression of specific functions that satisfy the differential equations. However, the vast majority of the ordinary differential equations, unfortunately, are not explicitly solvable so that we are unable to find specific expressions that solve the differential equations.

In the latter case, thanks to Sturm and Liouville's contributions, the Sturm-Liouville Oscillation Theory can be utilized to investigate qualitative properties of the solutions to some of the differential equations that we cannot solve explicitly. In particular, we are well-informed of the number and the location of the zeros of the solutions.

The Sturm-Liouville Oscillation Theory has applications in functional analysis and the analysis of operators. For certain matrices M such as the finite $n \times n$ Jacobi matrices J_N defined by

$$J_n = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & b_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & \cdots & a_{n-1} & b_n \end{bmatrix},$$

one can construct differential equations which have, as solutions, the characteristic polynomial of M. The zeros of the solutions of the differential equations are precisely the eigenvalues of M. Therefore, the Sturm-Liouville Oscillation Theory also provides information about the eigenvalues of the matrix M.

Now, we state the Sturm Oscillation Theorem using the difference equations approach:

Theorem 1.1. *(Sturm Oscillation Theorem). Let $P_1, P_2, \cdots, P_n$ be a sequence of monic orthogonal polynomials associated with the Jacobi matrix J. If $P_\ell(x_0) \neq 0$, $\ell = 0, 1, \cdots, n$, then the number of eigenvalues of J_n above x_0 is*

$$\#\{j \,|\, 0 \leq j \leq n-1 \textit{ so that } \operatorname{sgn}(P_{j+1}(x_0)) \neq \operatorname{sgn}(P_j(x_0))\}.$$

Using the same approach, the Sturm Comparison Theorem can be stated as:

Theorem 1.2. *(Sturm Comparison Theorem) If u and v are linearly independent solutions of*

$$x_0 u_n = a_{n+1}u_{n+1} + b_{n+1}u_n + a_n u_{n-1},$$

v has a sign change between every pair of sign changes of u.

Finally, we state the Sturm Comparison Theorem using the differential equations approach:

Theorem 1.3. *Consider a singular endpoint of equation*

$$(py')' + qy = 0 \quad on \quad J, \quad 1/p,\, q \in L^1_{loc}(J, \mathbb{R}), \quad p > 0 \ a.e. \quad on \quad J = (a,b). \quad (1.5)$$

Now, consider a comparison equation

$$(Pz')' + Qz = 0 \quad on \quad J. \tag{1.6}$$

Assume $1/P$, $Q \in L^1_{loc}(J, \mathbb{R})$ *and satisfy*

$$Q \geq q, \quad 0 < P \leq p \quad on \quad J. \tag{1.7}$$

Suppose y *is a non-trivial solution of* (1.5) *satisfying* $y(c) = 0 = y(d)$ *for some* $c, d \in J$, $c < d$. *Then every solution of* (1.6) *has a zero in the closed interval* $[c, d]$.

2 First Order Systems

2.1 Existence and Uniqueness of Solutions

Definition 2.1. (Contraction Mapping). A contraction mapping, or contraction, on a metric space (M, d) is a function f from M to itself, with the property that there is some real number $k < 1$ such that for all x and y in M,

$$d(f(x), f(y)) \leq k\, d(x, y).$$

The smallest such value of k is called the *Lipschitz constant* of f. Contractive maps are sometimes called Lipschitzian maps. If the above condition is satisfied for $k \leq 1$, then the mapping is said to be non-expansive.

The following theorem is a classic result for first-order systems stated in Zettl (2005). Zettl also provides the outline for two methods of proving the theorem.

Theorem 2.2. *(Existence and Uniqueness). Let* J *be any integral, open, closed, half open, bounded or unbounded; let* $n, m \in \mathbb{N}$. *If*

$$P \in M_n(L^1_{loc}(J, \mathbb{C})) \tag{2.1}$$

and

$$F \in M_{n,M}(L^1_{loc}(J, \mathbb{C})) \tag{2.2}$$

then every initial value problem (IVP)

$$Y' = PY + F \tag{2.3}$$

$$Y(u) = C,\ u \in J,\ C \in M_{n,m}(\mathbb{C}) \tag{2.4}$$

has a unique solution defined on all of J. Furthermore, if C, P, F are all real-valued, then there is a unique real valued solution.

Proof. (Method 1: Defining a contraction mapping on a Banach space)

First, we note that if Y is a solution of the IVP (2.3), (2.4) then we integrate both sides of (2.3), which yields

$$Y(t) = C + \int_u^t (PY + F),\ t \in J. \tag{2.5}$$

Conversely, every solution of the integral equation (2.5) has a unique solution on $[u, c]$ if $c > u$ and on $[c, u]$ if $c < u$. Assume $c > u$. Let

$$B = \{Y : [u, c] \to M_{n,m}(\mathbb{C}),\ Y \text{ continuous}\}.$$

Now, we need to define a contraction mapping T on the Banach space B such that $\|Tx\| \leq k\|x\|$, where $k < 1$. Then, we are able to apply the contraction mapping principle later in order to show that T has only one unique fixed point.

Following Bielecki (1956), we define the norm of any function $Y \in B$ to be

$$\|Y\| = \sup\{\exp\left(-K \int_u^t |P(s)|ds\right) |Y(t)|,\ t \in [u, c]\}, \tag{2.6}$$

where K is a fixed positive constant $K > 1$. It is easy to see that with this norm B is a Banach space since the Banach space is a metric space with norm. Let the operator $T : B \to B$ be defined by

$$(TY)(t) = C + \int_u^t (PY + F)(s)ds,\ t \in [u, c], Y \in B. \tag{2.7}$$

Then for $Y, Z \in B$ we have

$$TY(t) - (TZ)(t) = \int_u^t P(s)(Y(s) - Z(s))ds$$

so that

$$|TY(t) - (TZ)(t)| \leq \int_u^t |P(s)||Y(s) - Z(s)|ds$$

Now, let us look at $\exp\left(-K\int_u^t |P(s)|ds\right)|(TY)(t) - (TZ)(t)|$. By the above inequality, we have

$$\begin{aligned}
& \exp\left(-K\int_u^t |P(s)|ds\right)|(TY)(t) - (TZ)(t)| \\
\leq\ & \exp\left(-K\int_u^t |P(s)|ds\right)\int_u^t |P(s)||Y(s) - Z(s)|ds \\
=\ & \int_u^t \exp\left(-K\int_u^t |P(r)|dr\right)|P(s)||Y(s) - Z(s)|ds \\
=\ & \int_u^t \exp\left(-K\int_u^s |P(r)|dr\right)\exp\left(-K\int_s^t |P(r)|dr\right)|P(s)||Y(s) - Z(s)|ds \\
=\ & \int_u^t \exp\left(-K\int_s^t |P(r)|dr\right)|P(s)|\exp\left(-K\int_u^s |P(r)|dr\right)|Y(s) - Z(s)|ds \\
=\ & \int_u^t \exp\left(-K\int_s^t |P(r)|dr\right)|P(s)|\|Y - Z\|ds \\
=\ & \|Y - Z\|\int_u^t |P(s)|\exp\left(-K\int_s^t |P(r)|dr\right)ds.
\end{aligned}$$

Let $w(s) = -\int_s^t |P(r)|dr$ and thus by the Fundamental Theorem of Calculus, $dw(s) = |P(s)|ds$. Then by substitution we have

$$\begin{aligned}
& \int_u^t |P(s)|\exp\left(-K\int_u^t |P(r)|dr\right)ds \\
=\ & \int_{-\int_s^u |P(r)|dr}^{0} \exp(-Kw(s))dw(s) \\
=\ & \int_0^{\int_s^u |P(r)|dr} \exp(-Kw(s))dw(s) \\
=\ & \left[-\frac{\exp(-Kw)}{K}\right]_{w=0}^{w=\int_s^u |P(r)|dr} \\
=\ & \frac{1}{K} - \frac{1}{K}\exp\left(-K\int_s^u |P(r)|dr\right) \\
\leq\ & \frac{1}{K}.
\end{aligned}$$

Hence, we have

$$\begin{aligned} & \exp\left(-K\int_u^t |P(s)|ds\right)|(TY)(t)-(TZ)(t)| \\ \leq \;& \|Y-Z\|\int_u^t |P(s)|\exp\left(-K\int_s^t |P(r)|dr\right)ds \\ \leq \;& \frac{1}{K}\|Y-Z\|. \end{aligned}$$

Therefore, since by definition,

$$\|TY-TZ\| = \sup\{\exp\left(-K\int_u^t |P(s)|ds\right)|(TY)(t)-(TZ)(t)|,\ t\in[u,c]\},$$

and $\frac{1}{K}\|Y-Z\|$ is also an upper bound, we have

$$\|TY-TZ\| \leq \frac{1}{K}\|Y-Z\|.$$

From the contraction mapping principle in Banach space it follows that the map T has a unique fixed point and therefore the IVP (2.3), (2.4) has a unique solution on $[u,c]$. The proof for the case $c<u$ is similar; in this case the norm of B is modified to

$$\|Y\| = \sup\{\exp\left(K\int_u^t |P(s)|ds\right)|Y(t)|,\ t\in[c,u]\},$$

Since there is a unique solution on every compact subinterval $[u,c]$ and $[c,u]$ for $c\in J$, $c\neq u$ it follows that there is a unique solution on J. To establish the furthermore part take the Banach space of real-valued functions, that is,

$$B=\{Y:[u,c]\to M_{n,m}(\mathbb{R}),\ Y \text{ continuous}\}.$$

and proceed similarly. This completes the first proof. □

Proof. (Method 2: Standard successive approximations proof)

For the second proof we construct a solution of (2.5) by successive approximations. Define

$$Y_0(t)=C,\ Y_{n+1}(t)=C+\int_u^t (PY_n+F),\ t\in J,\ n=0,1,2,\cdots \tag{2.8}$$

Then Y_n is a continuous function on J for each $n \in \mathbb{N}$. We show that the sequence $\{Y_n : n \in \mathbb{N}\}$ converges to a function Y uniformly on each compact subinterval of J and that the limit function Y is the unique solution of the integral equation (2.5) and hence also of the IVP (2.3), (2.4). Choose $b \in J$, $b > u$ and define

$$p(t) = \int_u^t |P(s)|ds,\ t \in J; B_n(t) = \max_{u \le s \le t} |Y_{n+1}(s) - Y_n(s)|,\ u \le t \le b. \tag{2.9}$$

Then,

$$Y_{n+1}(t) - Y_n(t) = \int_u^t P(s)[Y_n(s) - Y_{n-1}(s)]ds,\ t \in J,\ n \in \mathbb{N}. \tag{2.10}$$

From this we get

$$|Y_2(t) - Y_1(t)| \le B_0(t) \int_u^t |P(s)|ds = B_0(t)p(t) \le B_0(t)p(b),\ u \le t \le b. \tag{2.11}$$

$$\begin{aligned} |Y_3(t) - Y_2(t)| &= \int_u^t |P(s)||Y_2(t) - Y_1(t)|ds \le \int_u^t |P(s)|B_0(s)p(s)ds \\ &\le B_0(t) \int_u^t |P(s)|p(s)ds \le B_0(b)\frac{p^2(t)}{2!} \\ &\le B_0(b)\frac{p^2(b)}{2!},\ u \le t \le b. \end{aligned}$$

From this and mathematical induction we get

$$|Y_{n+1}(t) - Y_n(t)| \le B_0(b)\frac{p^n(b)}{n!},\ u \le t \le b.$$

Hence for any $k \in \mathbb{N}$, by the triangle inequality,

$$\begin{aligned} |Y_{n+k+1}(t) - Y_n(t)| &\le |Y_{n+k+1}(t) - Y_{n+k}(t)| + |Y_{n+k}(t) - Y_{n+k-1}(t)| + \\ &\quad \cdots + |Y_{n+1}(t) - Y_n(t)| \\ &\le B_0(b)\frac{p^n(b)}{n!}[1 + \frac{p(b)}{n+1} + \frac{p^2(b)}{(n+2)(n+1) + \cdots}]. \end{aligned}$$

Choose m large enough so that $p(b)/(n+1) \le 1/2$, then $p^2(b)/((n+2)(n+1)) \le 1/4$, etc. when $n > m$ and the term in brackets is bounded above by 2. It follows that the sequence $\{Y_n : n \in \ N_0\}$ converges uniformly, say to Y, on $[u, b]$. From this it follows that Y satisfies the integral equation (2.5) and hence also the IVP (2.3) and (2.4) on $[u, b]$.

To show that Y is the unique solution assume Z is another one; then Z is continuous and therefore $|Y - Z|$ is bounded, say by $M > 0$ on $[u, b]$. Then

$$|Y(t) - Z(t)| = \left| \int_u^t P(s)[Y(s) - Z(s)]ds \right| \leq M \int_u^t |P(s)|ds \leq Mp(t), \ u \leq t \leq b.$$

Now proceeding as above we get

$$|Y(t) - Z(t)| \leq M \frac{p^n(t)}{n!} \leq M \frac{p^n(b)}{n!}, \ u \leq t \leq b, \ n \in \mathbb{N}.$$

Note that as n approaches ∞, $M \frac{p^n(b)}{n!}$ approaches 0 so that therefore, $Y = Z$ on $[u, b]$. There is a similar proof for the case when $b < u$. This completes the second proof. □

2.2 Properties of the First Order Systems

Theorem 2.3. *(The Gronwall Inequality)*

(i) (The "right" Gronwall inequality) Let $J = [a, b]$. Assume g in $L^1(J, \mathbb{R})$ with $g \geq 0$ a.e., f real valued and continuous on J. If y is continuous, real valued, and satisfies

$$y(t) \leq f(t) + \int_a^t g(s)y(s)ds, \quad a \leq t \leq b, \tag{2.12}$$

then

$$y(t) \leq f(t) + \left(\int_a^t f(s)g(s) \exp \left(\int_s^t g(u)du \right) ds \right), \quad a \leq t \leq b, \tag{2.13}$$

For the special case when $f(t) = c$, a constant, we get

$$y(t) \leq c \exp \left(\int_a^t g(s)ds \right), \quad a \leq t \leq b. \tag{2.14}$$

For the special case when f is nondecreasing on $[a, b]$ we get

$$y(t) \leq f(t) \exp \left(\int_a^t g(s)ds \right), \quad a \leq t \leq b. \tag{2.15}$$

(ii) *(The "left" Gronwall inequality) Let $J = [a, b]$. Assume g in $L^1(J, \mathbb{R})$ with $g \geq 0$ a.e., f real valued and continuous on J. If y is continuous, real valued, and satisfies*

$$y(t) \leq f(t) + \int_t^b g(s)y(s)ds, \quad a \leq t \leq b, \tag{2.16}$$

then

$$y(t) \leq f(t) + \left(\int_t^b f(s)g(s)\exp\left(\int_t^s g(u)du\right) ds\right), \quad a \leq t \leq b, \tag{2.17}$$

For the special case when $f(t) = c$, a constant, we get

$$y(t) \leq c\exp\left(\int_t^b g(s)ds\right), \quad a \leq t \leq b. \tag{2.18}$$

For the special case when f is nondecreasing on $[a, b]$ we get

$$y(t) \leq f(t)\exp\left(\int_t^b g(s)ds\right), \quad a \leq t \leq b. \tag{2.19}$$

Proof. For part (i) let $z(t) = \displaystyle\int_a^t gy$, $t \in J$ and note that

$$z' = gy \leq g\left(f + \int_a^t gy\right) = gf + gz \implies z' - gz \leq gf \ \ a.e.$$

Hence, we have

$$\begin{aligned}
& \exp\left(-\int_a^s g(u)du\right)[z'(s) - g(s)z(s)] \\
= \ & \exp\left(-\int_a^s g(u)du\right)z'(s) - \exp\left(-\int_a^s g(u)du\right)g(s)z(s) \\
= \ & \left[\exp\left(-\int_a^s g(u)du\right)z(s)\right]' \leq g(s)f(s)\exp\left(-\int_a^s g(u)du\right), \quad a \leq s \leq b.
\end{aligned}$$

Integrating both sides of the inequality above from a to t, we get

$$\exp\left(-\int_a^t g(u)du\right)z(t) \leq \int_a^t g(s)f(s)\exp\left(-\int_a^s g(u)du\right)ds, \quad a \leq t \leq b.$$

From (2.12) and the above line we obtain

$$
\begin{aligned}
y(t) &\leq f(t) + z(t) \leq f(t) + \exp\left(\int_a^t g(u)du\right) \exp\left(-\int_a^t g(u)du\right) z(t) \\
&\leq f(t) + \exp\left(\int_a^t g(u)du\right) \int_a^t g(s)f(s) \exp\left(-\int_a^s g(u)du\right) ds \\
&= f(t) + \int_a^t g(s)f(s) \exp\left(\int_a^t g(u)du\right) \exp\left(-\int_a^s g(u)du\right) ds \\
&= f(t) + \left(\int_a^t f(s)g(s) \exp\left(\int_s^t g(u)du\right) ds\right), \quad a \leq t \leq b.
\end{aligned}
$$

This concludes the proof of (2.13). The two special cases follow from (2.13).

If $f(t) = c$, where c is a constant, then by (2.13), we have

$$
\begin{aligned}
y(t) &\leq f(t) + \left(\int_a^t f(s)g(s) \exp\left(\int_s^t g(u)du\right) ds\right) \\
&= c + \left(\int_a^t cg(s) \exp\left(\int_s^t g(u)du\right) ds\right) \\
&= c - c\exp\left(\int_s^t g(s)ds\right)\Bigg|_{s=a}^{s=t} = c\exp\left(\int_a^t g(s)ds\right), \quad a \leq t \leq b.
\end{aligned}
$$

If f is a nondecreasing on $[a,b]$, then by (2.13), we have

$$
\begin{aligned}
y(t) &\leq f(t) + \left(\int_a^t f(s)g(s) \exp\left(\int_s^t g(u)du\right) ds\right) \\
&\leq f(t) + f(t)\left(\int_a^t g(s) \exp\left(\int_s^t g(u)du\right) ds\right) \\
&= f(t) - f(t)\exp\left(\int_s^t g(s)ds\right)\Bigg|_{s=a}^{s=t} \\
&= f(t)\exp\left(\int_a^t g(s)ds\right), \quad a \leq t \leq b.
\end{aligned}
$$

For part (ii) let $z(t) = \displaystyle\int_t^b gy$, $t \in J$ and note that

$$
z' = -gy \geq -g\left(f + \int_t^b gy\right) = -gf - gz \implies z' + gz \geq -gf \quad a.e.
$$

Hence, we have

$$\begin{aligned}
& \exp\left(-\int_s^b g(u)du\right)[z'(s)+g(s)z(s)] \\
= & \exp\left(-\int_s^b g(u)du\right)z'(s)+\exp\left(-\int_s^b g(u)du\right)g(s)z(s) \\
= & \left[\exp\left(-\int_s^b g(u)du\right)z(s)\right]' \geq -g(s)f(s)\exp\left(-\int_s^b g(u)du\right), \quad a \leq s \leq b.
\end{aligned}$$

Integrating both sides of the inequality above from t to b, we get

$$\begin{aligned}
& -\exp\left(-\int_t^b g(u)du\right)z(t) \geq -\int_t^b g(s)f(s)\exp\left(-\int_s^b g(u)du\right)ds, \quad a \leq t \leq b \\
\Longrightarrow & \exp\left(-\int_t^b g(u)du\right)z(t) \leq \int_t^b g(s)f(s)\exp\left(-\int_s^b g(u)du\right)ds, \quad a \leq t \leq b.
\end{aligned}$$

From (2.16) and the above line we obtain

$$\begin{aligned}
y(t) & \leq f(t)+z(t) \leq f(t)+\exp\left(\int_t^b g(u)du\right)\exp\left(-\int_t^b g(u)du\right)z(t) \\
& \leq f(t)+\exp\left(\int_t^b g(u)du\right)\int_t^b g(s)f(s)\exp\left(-\int_s^b g(u)du\right)ds \\
& = f(t)+\int_t^b g(s)f(s)\exp\left(\int_t^b g(u)du\right)\exp\left(-\int_s^b g(u)du\right)ds \\
& = f(t)+\left(\int_t^b f(s)g(s)\exp\left(\int_t^s g(u)du\right)ds\right), \quad a \leq t \leq b.
\end{aligned}$$

This concludes the proof of (2.17). The two special cases follow from (2.17).

If $f(t)=c$, where c is a constant, then by (2.17), we have

$$\begin{aligned}
y(t) & \leq f(t)+\left(\int_t^b f(s)g(s)\exp\left(\int_t^s g(u)du\right)ds\right) \\
& = c+\left(\int_t^b cg(s)\exp\left(\int_t^s g(u)du\right)ds\right) \\
& = c-c\exp\left(\int_t^s g(s)ds\right)\Bigg|_{s=t}^{s=b} \\
& = c\exp\left(\int_t^b g(s)ds\right), \quad a \leq t \leq b.
\end{aligned}$$

If f is a nondecreasing on $[a,b]$, then by (2.17), we have

$$\begin{aligned} y(t) &\leq f(t) + \left(\int_t^b f(s)g(s)\exp\left(\int_t^s g(u)du \right) ds \right) \\ &\leq f(t) + f(t)\left(\int_t^b g(s)\exp\left(\int_t^s g(u)du \right) ds \right) \\ &= f(t) - f(t)\exp\left(\int_t^s g(s)ds \right)\Bigg|_{s=t}^{s=b} \\ &= f(t)\exp\left(\int_t^b g(s)ds \right), \quad a \leq t \leq b \end{aligned}$$

and this completes the proof. □

Theorem 2.4. *(Abel's Formula) Let $J = (a,b)$, and assume that $P \in M_n(L^1_{loc}(J,\mathbb{C}))$. If Y is an $n \times n$ matrix solution of*

$$Y' = PY \text{ on } J,$$

then for any $u, t \in J$, we have

$$(\det Y)(t) = (\det Y)(u)\exp\left(\int_u^t \operatorname{trace} P(s)ds \right).$$

Proof. First, we want to show that $y = \det Y$ satisfies that first order scalar equation

$$y' - py = 0,$$

where $p = \operatorname{trace} P$.

Let y_{ij} denote the ij^{th} entry of $Y(t)$. Note that by the definition of the determinant,

$$y = \det Y = \sum_{\sigma \in S_n} \epsilon(\sigma) y_{1\sigma(1)} \cdots y_{n\sigma(n)},$$

where $\sigma(i)$ is in the symmetry group for every i, and hence

$$y' = \sum_{i=1}^n \sum_{\sigma \in S_n} \epsilon(\sigma) y_{1\sigma(1)} \cdots y'_{i\sigma(i)} \cdots y_{n\sigma(n)}$$

by the product rule.

By the definition of the determinant again, it follows that

$$y' = \frac{dy}{dt} = \frac{d}{dt}(\det Y(t)) = \sum_{i=1}^{n} \det \begin{bmatrix} y_{11}(t) & \cdots & y_{1n}(t) \\ \vdots & \ddots & \vdots \\ y'_{i1}(t) & \cdots & y'_{in}(t) \\ \vdots & \ddots & \vdots \\ y_{n1}(t) & \cdots & y_{nn}(t) \end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} y'_{ij}(t) M_{ij}, \quad (2.20)$$

where M_{ij} is the corresponding matrix when the i^{th} row and j^{th} column of the previous matrix removed.

Since $Y' = PY$, by the multiplication of matrices, we know that

$$y'_{ij}(t) = \sum_{k=1}^{n} p_{ik}(t) y_{kj}(t).$$

Thus, the i^{th} row of the equation (2.20) can be replaced by

$$(y'_{i1}(t), \cdots, y'_{in}(t)) = \sum_{k=1}^{n} p_{ik}(y_{k1}(t), \cdots, y_{kn}(t)).$$

Recall the property of the determinant that if matrix B is obtained by multiplying one row or column of matrix A by some number c, then $\det B = c \det A$. Plugging the equality above into equation (2.20) and by the above property of the determinant, we have

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(\det Y(t)) \\ &= \sum_{i=1}^{n} \det \begin{bmatrix} y_{11}(t) & \cdots & y_{1n}(t) \\ \vdots & \ddots & \vdots \\ y'_{i1}(t) & \cdots & y'_{in}(t) \\ \vdots & \ddots & \vdots \\ y_{n1}(t) & \cdots & y_{nn}(t) \end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{n} y'_{ij}(t) M_{ij} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} p_{ik}(t) y_{kj}(t) M_{ij} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}(t) \sum_{j=1}^{n} y_{kj}(t) M_{ij}, \end{aligned}$$

where the last term

$$\sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}(t)\sum_{j=1}^{n} y_{kj}(t)M_{ij} = \sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}(t)\det\begin{bmatrix} y_{11}(t) & \cdots & y_{1n}(t) \\ \vdots & \ddots & \vdots \\ y_{k1}(t) & \cdots & y_{kn}(t) \\ \vdots & \ddots & \vdots \\ y_{n1}(t) & \cdots & y_{nn}(t) \end{bmatrix}.$$

Note that the row written in the middle denotes the generic k^{th} row.

When $k \neq i$, then the above matrix will have two rows that are exactly the same and will also miss the i^{th} row. Instead, when $i = k$, the above matrix will be exactly $Y(t)$.

Note that if two rows of a matrix are the same, then the determinant of that matrix will be 0. Here is the explanatin: the determinant should be the additive inverse of itself in this case, since interchanging the rows (or columns) does not change the absolute value of the determinant, but reverses the sign of the determinant. The only number for which this is possible is when it is equal to 0.

Hence, the last term in the last equation can be simplified to

$$\begin{aligned}\sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}(t)\det\begin{bmatrix} y_{11}(t) & \cdots & y_{1n}(t) \\ \vdots & \ddots & \vdots \\ y_{k1}(t) & \cdots & y_{kn}(t) \\ \vdots & \ddots & \vdots \\ y_{n1}(t) & \cdots & y_{nn}(t) \end{bmatrix} &= \sum_{i=1}^{n} p_{ii}(t)\det Y(t) = \det Y(t)\sum_{i=1}^{n} p_{ii}(t) \\ &= y\cdot \operatorname{trace} P = py.\end{aligned}$$

On the left-hand side, we have y', and we have shown above that the right-hand side can be simplified as py. Hence, we have $y' = py$, where $y' = \det Y$ and $p = \operatorname{trace} P$.

With $y' = py$, for any $u, t \in J$, we have

$$\frac{1}{y}dy = p \Longrightarrow \int_u^t \frac{1}{y}dy = \int_u^t p \Longrightarrow \ln y(t) - \ln y(u) = \int_u^t p(s)ds.$$

Exponentiating both sides, we have

$$y(t) = y(u)\exp\left(\int_u^t p(s)ds\right),$$

from which it follows that

$$(\det Y)(t) = (\det Y)(u) \exp\left(\int_u^t \operatorname{trace} P(s)ds\right),$$

which completes the proof. □

3 Second Order Scalar Equations

In this section, we study the initial value problems (IVP) consisting of the second order scalar equation

$$-(py')' + qy = f \text{ on } J \tag{3.1}$$

together with the initial conditions

$$y(c) = h, \quad (py')(c) = k, \quad c \in J, \quad h, k \in \mathbb{C} \tag{3.2}$$

where

$$J = (a,b), \quad -\infty \leq a < b \leq +\infty, \quad \frac{1}{p},\, q,\, f : J \to \mathbb{C}. \tag{3.3}$$

3.1 Existence and Uniqueness of Solutions

Definition 3.1. (Solution). By a solution of equation (5.1) we mean a function $y : J \to \mathbb{C}$ such that y and $y^{[1]} = py'$ are absolutely continuous on each compact subinterval of J and the equation is satisfied a.e. on J. Given a solution y we refer to $y^{[1]}$ as its quasi-derivative to distinguish it from the classical derivative y'.

Note that the classical derivative of a solution y, in general, exists only almost everywhere but the quasi-derivative $y^{[1]} = (py')$ is absolutely continuous on all compact subintervals of J and thus exists and is continuous at each point of the open interval J.

Theorem 3.2. *Assume that*

$$1/p,\ q,\ f \in L^1_{loc}(J, \mathbb{C}). \tag{3.4}$$

Then every initial value problem (IVP) (5.1), (5.2), (5.3) *has a solution defined on all of J and this solution is unique. Moreover, if all the data p, q, f, h, k is real, then there is a unique real-valued solution on J.*

Proof. First, let us construct the following matrices:

$$P = \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 \\ -f \end{bmatrix}, \ Y = \begin{bmatrix} y \\ py' \end{bmatrix}. \tag{3.5}$$

Then, we observe that

$$\begin{aligned} PY + F &= \begin{bmatrix} 0 & 1/p \\ q & 0 \end{bmatrix} \begin{bmatrix} y \\ py' \end{bmatrix} + \begin{bmatrix} 0 \\ -f \end{bmatrix} \\ &= \begin{bmatrix} y' \\ qy - f \end{bmatrix} \\ &= \begin{bmatrix} y' \\ (py')' \end{bmatrix} \\ &= Y'. \end{aligned}$$

Hence, the equation (5.1) is equivalent to the first order system

$$Y' = PY + F \text{ on } J, \tag{3.6}$$

in the sense that, given any scalar solution y of (5.1) the vector Y defined by (3.1) is a solution of the system (3.6) and conversely, given any vector solution Y of system (3.6) its top component y is a solution of (5.1). Theorem 3.2 follows from this system representation and Theorem 2.2 of the previous section. □

In the theory of boundary value problems the spectral parameter λ and the weight function w play important roles; thus we also study the equation

$$-(py')' + qy = \lambda wy \text{ on } J \tag{3.7}$$

where

$$J = (a, b), \quad -\infty \leq a < b \leq +\infty, \quad \frac{1}{p}, \ q, \ w : L^1_{loc}(J, \mathbb{C}). \tag{3.8}$$

In this case, we use the following notation

$$y = y(\cdot, c, h, k, 1/p, q, w, \lambda),\ y^{[1]} = y^{[1]}(\cdot, c, h, k, 1/p, q, w, \lambda). \tag{3.9}$$

Definition 3.3. (Regular and Singular Endpoints). Let $J = (a, b), -\infty \leq a < b \leq \infty$, and consider the equation (5.1) with conditions (3.4).

The endpoint a is said to be regular (or equation (5.1) is regular at a) if

$$1/p, q, f \in L^1((a, d), \mathbb{C}) \tag{3.10}$$

for some $d \in J$; otherwise it is called singular. Similarly, the endpoint b is said to be regular if

$$1/p, q, f \in L^1((d, b), \mathbb{C}) \tag{3.11}$$

for some $d \in J$; otherwise it is called singular. Note that, given condition (3.4), if (3.10) and (3.11) hold for some $d \in J$, then they hold for any $d \in J$.

The following theorem is stated in Zettl (2005) without correct proof.

Theorem 3.4. *Let* (3.7),(3.8) *hold in* $\mathbb{R}$ *and assume that* $p > 0$ *a.e. on* J *and* λ *is real. Then the zeros of every nontrivial solution* y *of* (3.7) *are isolated in the interior of* J *and also at regular endpoint of* J*. If a nontrivial solution* y *has a zero at a regular endpoint of* J *then there is an appropriate one sided neighborhood of this endpoint in which* y *has no other zero. Thus only a singular endpoint of* J *can be an accumulation point of zeros of any nontrivial solution* y *of* (3.7).

Proof. Suppose that a and b are both regular endpoints of J. By definition, for any $k \in J$, we know that

$$1/p, q, f \in L^1((a, k), \mathbb{C})$$

and that

$$1/p, q, f \in L^1((k, b), \mathbb{C})$$

so that

$$1/p, q, f \in L^1((a, b), \mathbb{C}),$$

which implies that $1/p, q, f$ are all locally integrable on the interval (a, b).

First, we show that if y has zeros at $c, d \in (a, b)$ with $c < d$, then $(py')(h) = 0$ for some $h \in (c, d)$. Then, we have

$$0 = y(d) - y(c) = \int_c^d y' = \int_c^d \frac{1}{p}(py') = (py')(h) \int_c^d \frac{1}{p}$$

by the Mean Value Theorem for the Lebesgue integral (recall that (py') is continuous on J). Moreover, we are allowed to conduct the operations above because $1/p \in L^1((a, b), \mathbb{C})$.

Hence, either

$$(py')(h) = 0$$

or

$$\int_c^d \frac{1}{p} = 0$$

for the above equality to hold. However, we claim that it is impossible to have the latter: by our hypothesis that $p > 0$ a.e. in (a, b), we know that $1/p > 0$ a.e. in (a, b) so that

$$\int_c^d \frac{1}{p} > 0$$

since the integral of a positive function on some interval (c, d) is always positive. Hence, we know that it must be true that $(py')(h) = 0$.

Now, in order to prove the Theorem, let us suppose that there exists a sequence $\{t_n \in (a, b) : n \in N_0\}$ such that $t_n \to t_0$ and $y(t_n) = 0$ for all $n \in N_0$. That is, the sequence above is composed of zeros of y converging to t_0. Then, since y is continuous, we know that $y(t_0) = 0$.

Then, suppose that there exists another sequence $\{h_n \in (a, b) : n \in N_0\}$ such that $h_n \to t_0$ and $(py')(h_n) = 0$ for all $n \in N_0$. Such a sequence must exist since we have shown above that there exists an h between any two zeros of y such that $(py')(h) = 0$. Then, since $y^{[1]}$ is continuous in (a, b), we know that $(py')(t_0) = 0$. Hence, we have

$$y(t_0) = 0$$

and

$$y^{[1]}(t_0) = 0 \Longrightarrow y'(t_0) = 0.$$

However, $y(t_0) = 0$ and $y^{[1]}(t_0)$ implies that y is identically zero on J. Thus, $y = 0$ is the only solution by the uniqueness of initial value problem. Recall that we have assumed y to be a nontrivial solution. This contradiction completes the proof. □

Theorem 3.5. *(Sturm Separation Theorem). Let* (3.7),(3.8) *hold in* $\mathbb{R}$ *and assume that* $p > 0$ *a.e. on* J *and* λ *is real. Suppose that* y *and* z *are linearly independent solutions of* (3.7)*. Then* z *has a zero strictly between any two zeros of* y*.*

Proof. Suppose

$$y(c) = 0 = y(d)$$

with $c \neq d$. Since y and z are linearly independent, neither is the trivial solution and they have no common zero. By Theorem 3.4, we may assume that c and d are consecutive zeros of y and that $c < d$. Further, replacing y by $-y$ if necessary, we may assume that $y > 0$ on the open interval (c, d). Then

$$0 < y(t) - y(c) = \int_c^t \frac{1}{p}(py').$$

This imples that $(py')(c) > 0$. Otherwise, since py' is continuous and $(py')(c)$ is not zero, the assumption that $(py')(c) < 0$ implies that it is negative in some right neighborhood of c and this contradicts the above equation. Similarly we get that $(py')(d) < 0$.

Now, multiplying the equation for y by z and the equation for z by y and subtracting we get

$$0 = z(py')' - y(pz')' = [z(py') - y(pz')]'.$$

An integration yields

$$z(d)(py')(d) - y(c)(pz')(c) = 0.$$

Assume that z has no zero in (c, d). If $z > 0$ on (c, d) then the left hand side of the above equation is positive giving a contradiction. A similar contradiction is reached if $z < 0$, which completes the proof. □

3.1.1 Examples

Example 3.6. *Let us consider the equation*

$$-(py')' + qy = \lambda wy$$

on $J = (a,b)$, *where*

$$J = (a,b) = (0,1);\ \frac{1}{p(t)} = t^{-2}\sin\left(\frac{1}{t}\right), \textit{where } 0 < t < 1;\ q = 0;\ w = 0.$$

Then, we have

$$-(py')' = 0$$

First, we need to solve the differential equation. Since $-(py')' = 0$, we know that $(py')' = 0$. Thus, $py' = C$, where C is a constant. If $C = 0$, then $y' = 0$ so that one possible solution is $u(t) = 1$. If $C \neq 0$, then $y' = \dfrac{C}{p}$ so that

$$y = \int \frac{C}{p} = C\int \frac{1}{p} = C\int t^{-2}\sin\left(\frac{1}{t}\right)dt = C\cos\left(\frac{1}{t}\right).$$

Hence, the other possible solution is

$$v(t) = \cos\left(\frac{1}{t}\right).$$

Let us observe the solution $v(t)$. As $t \to 0$, $\dfrac{1}{t} \to \infty$ so that the solution has infinitely many zeros in any right neighborhood of 0, as shown below.

We claim that 0 must be a singular endpoint in this case. Here is the reason:

Let d be any point in $(0,1)$, then

$$\int_0^d \frac{1}{p(t)}dt = \int_0^d t^{-2}\sin\left(\frac{1}{t}\right)dt = \cos\left(\frac{1}{t}\right)\Bigg|_{t=0}^{t=d} = \cos\left(\frac{1}{d}\right) - \lim_{t\to 0}\cos\left(\frac{1}{t}\right),$$

which does not exist since the limit of $\cos\left(\dfrac{1}{t}\right)$ as $t \to 0$ does not exist.

Example 3.7. *Let us consider the equation*

$$-(py')' + qy = \lambda wy$$

on $J = (a,b)$, *where*

$$J = (a,b) = (k,1);\ k > 0;\ \frac{1}{p(t)} = t^{-2}\sin\left(\frac{1}{t}\right), \textit{where } k < t < 1;\ q = 0;\ w = 0.$$

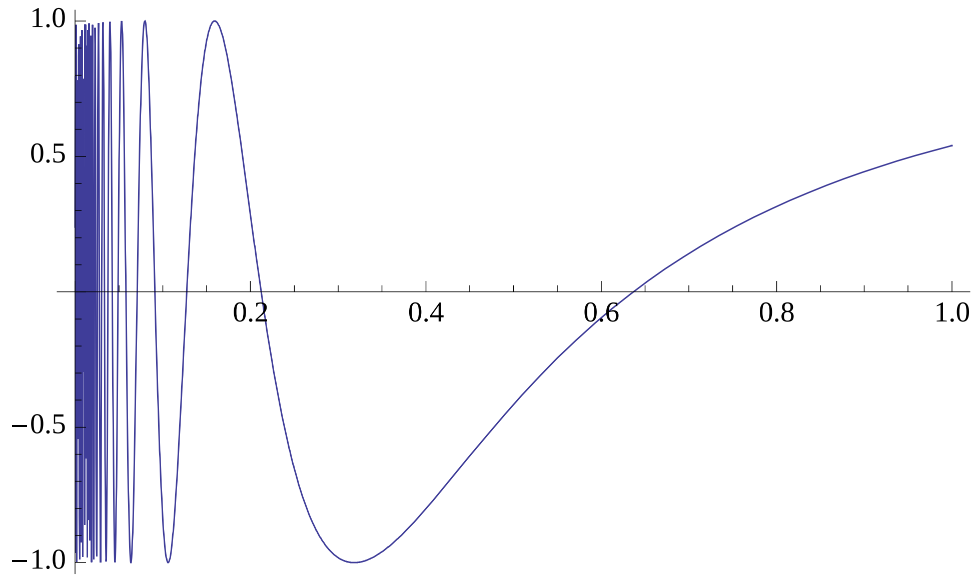

Figure 1: Example 3.6

Then, we have

$$-(py')' = 0$$

As shown in the previous example, we still have two solutions:

$$u(t) = 1, \quad v(t) = \cos\left(\frac{1}{t}\right).$$

Let us observe the solution $v(t)$. As $t \to k$, $\dfrac{1}{t} \to \dfrac{1}{k} < \infty$ and As $t \to 1$, $\dfrac{1}{t} \to 1 < \infty$. Hence, the solution has finitely many zeros in the interval $(k, 1)$, as shown below.

We claim that k must be a regular endpoint in this case. Here is the reason:

Let d be any point in $(k, 1)$, then

$$\int_k^d \frac{1}{p(t)} dt = \int_k^d t^{-2} \sin\left(\frac{1}{t}\right) dt = \cos\left(\frac{1}{t}\right)\Bigg|_{t=k}^{t=d} = \cos\left(\frac{1}{d}\right) - \cos\left(\frac{1}{k}\right) < \infty,$$

so that $1/p \in L^1((k, d), \mathbb{C})$.

Example 3.8. *Let us consider the equation*

$$-(py')' + qy = \lambda w y$$

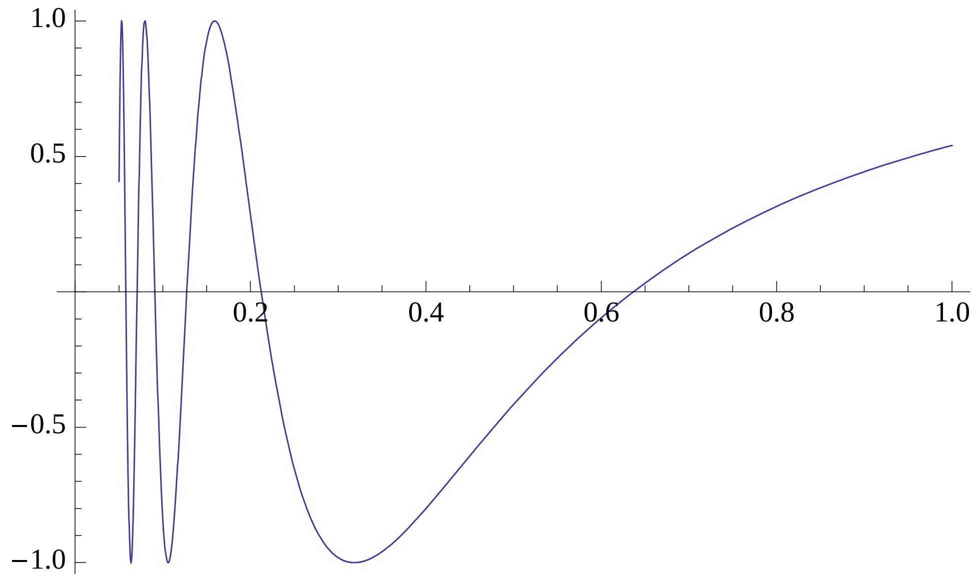

Figure 2: Example 3.7

on $J = (a, b)$, *where*

$$J = (a,b) = (0,1);\ k > 0;\ \frac{1}{p(t)} = t^{-3/2}\sin\left(\frac{1}{t}\right) + (1/2)t^{-1/2}\cos\left(\frac{1}{t}\right);\ q = 0;\ w = 0.$$

Then, we have

$$-(py')' = 0$$

Using the same method as the above, the solutions to this differential equation are:

$$u(t) = 1, \quad v(t) = \sqrt{t}\cos\left(\frac{1}{t}\right).$$

Let us observe the solution $v(t)$. As $t \to 0$, $\frac{1}{t} \to \infty$ so that the solution has infinitely many zeros in any right neighborhood of 0, as shown below. Thus, 0 is an accumulation point and must be a singular endpoint.

To check that 0 is a singular endpoint is somewhat trickier than in the previous cases. Here, we must distinguish between functions that are Riemann integrable and Lebesgue integrable.

We claim that $1/p(t)$ is Riemann integrable on the interval $(0, d)$, where $d \in (0, 1)$, but not Lebesgue integrable. To calculate the Riemann integral, we must use improper Riemann integral in this case because the function is not defined at 0.

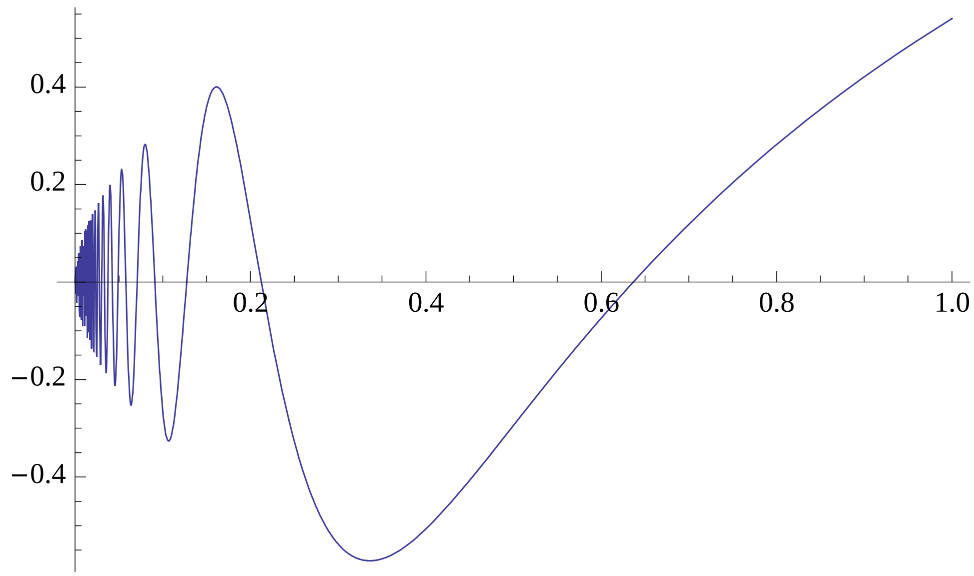

Figure 3: Example 3.8

Let d be any point in $(0,1)$, then

$$\begin{aligned}\int_0^d \frac{1}{p(t)}dt &= \int_0^d t^{-3/2}\sin\left(\frac{1}{t}\right) + (1/2)t^{-1/2}\cos\left(\frac{1}{t}\right)dt = \sqrt{t}\cos\left(\frac{1}{t}\right)\Big|_{t=0}^{t=d}\\ &= \sqrt{d}\cos\left(\frac{1}{d}\right) - \lim_{t\to 0}\sqrt{t}\cos\left(\frac{1}{t}\right)\\ &= \sqrt{d}\cos\left(\frac{1}{d}\right) - 0\\ &= \sqrt{d}\cos\left(\frac{1}{d}\right).\end{aligned}$$

Hence, $1/p(t)$ is indeed improperly Riemann integrable on $(0,d)$.

However, note that

$$\int_0^d \left|\frac{1}{p(t)}\right| dt \to \infty$$

since there is no longer the cancelation between the positive values and negative values. This implies that the function $1/p(t)$ is not Lebesgue integrable on $(0,d)$. By definition, 0 must be a singular endpoint.

Example 3.9. *Let us consider the equation*

$$-(py')' + qy = \lambda wy$$

on $J = (a, b)$, where

$$J = (0, a);\ a < \infty;\ \frac{1}{p(t)} = \frac{1}{t^{\alpha}}, \textit{ where } k < t < 1 \textit{ and } \alpha \in [-\infty, 1);\ q = 0;\ w = 0.$$

Then, we have

$$-(py')' = 0$$

First, we need to solve the differential equation. Since $-(py')' = 0$, we know that $(py')' = 0$. Thus, $py' = C$, where C is a constant. If $C = 0$, then $y' = 0$ so that one possible solution is $u(t) = 1$. If $C \neq 0$, then $y' = \dfrac{C}{p}$ so that

$$y = \int \frac{C}{p} = C \int \frac{1}{p} = C \int \frac{1}{t^{\alpha}} dt = \frac{C}{-\alpha + 1} \frac{1}{t^{\alpha - 1}}.$$

Hence, the other possible solution is

$$v(t) = \frac{1}{(-\alpha + 1)t^{\alpha - 1}} = \frac{t^{1-\alpha}}{(-\alpha + 1)}.$$

Let us observe the solution $v(t)$. Recall that $\alpha < 1$ so that $1 - \alpha > 0$. As $t \to 0$, $t^{1-\alpha} \to 0$. Since $v(t)$ is strictly increasing, we know that v does not have any zeros on the interval $(0, a)$ with $a < \infty$.

Example 3.10. *Let us consider the equation*

$$-(py')' + qy = \lambda wy$$

on $J = (a, b)$, where

$$J = (0, a);\ a < \infty;\ \frac{1}{p(t)} = \frac{1}{t^{\alpha}}, \textit{ where } k < t < 1 \textit{ and } \alpha \in [1, \infty);\ q = 0;\ w = 0.$$

Then, we have

$$-(py')' = 0$$

As we have shown in the previous example, one possible solution is $u(t) = 1$ and the other possible solution is

$$v(t) = \frac{1}{(-\alpha + 1)t^{\alpha - 1}} = \frac{t^{1-\alpha}}{(-\alpha + 1)}.$$

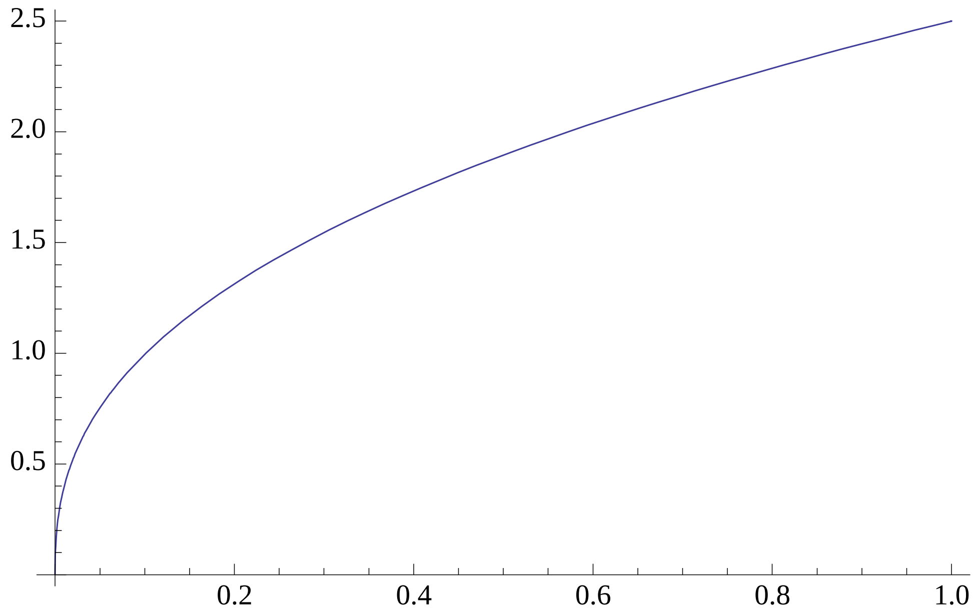

Figure 4: Example 3.9

Let us observe the solution $v(t)$. Recall that $\alpha \geq 1$ so that $\alpha - 1 \geq 0$. As $t \to 0$, $t^{\alpha-1} \to 0$ so that $v(t) \to -\infty$. Since $v(t)$ is strictly increasing and $v(t) < 0$, we know that v does not have any zeros on the interval $(0, a)$ with $a < \infty$.

However, we claim that 0 is be a singular endpoint in this case. Here is the reason:

Let d be any point in $(0, a)$, then

$$\int_0^d \frac{1}{p(t)} dt = \int_0^d \frac{1}{t^\alpha} dt = \frac{t^{1-\alpha}}{(-\alpha+1)}\bigg|_{t=0}^{t=d} = \frac{d^{1-\alpha}}{(-\alpha+1)} - \lim_{t\to 0} \frac{t^{1-\alpha}}{(-\alpha+1)},$$

which does not exist since the limit of $\dfrac{t^{1-\alpha}}{(-\alpha+1)}$ as $t \to 0$ does not exist when $\alpha \geq 1$.

3.2 Differentiable Dependence

Now, we will introduce the concept of the Fréchet derivative, which is a derivative defined on Banach spaces. Named after Maurice Fréchet, it is commonly used to formalize the concept of the functional derivative used widely in the calculus of variations. That is, each derivate is regarded as a bounded linear operator T. Intuitively, it generalizes the idea of linear approximation from functions of one variable to functions on Banach

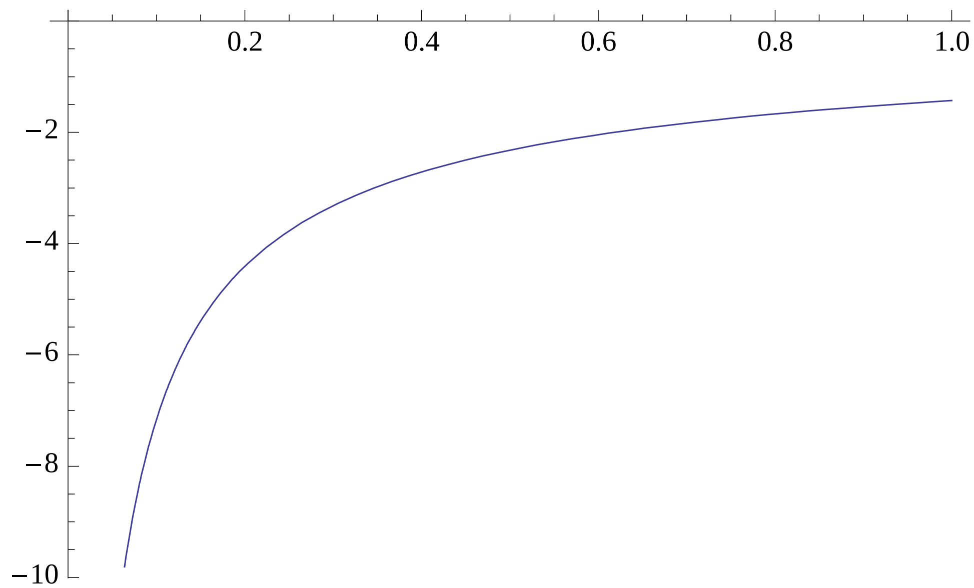

Figure 5: Example 3.10

spaces. The Fréchet derivative should be contrasted to the more general Gâteaux derivative which is a generalization of the classical directional derivative.

Definition 3.11. (Fréchet derivative in Banach spaces) Let V and W be Banach spaces, and $U \subseteq V$ be an open subset of V. A function $f : U \to W$ is called Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $T : V \to W$ such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - T(h)\|_W}{\|h\|_V} = 0.$$

Theorem 3.12. *(Differentiable dependence on the equation) Let conditions* (3.7), (3.8) *hold, and let K be a compact subinterval of J. Fix $t, c \in K$, $h, k \in \mathbb{C}$. Then the maps $q \to y(t,q)$, $q \to y^{[1]}(t,q)$, $w \to y(t,w)$, $w \to y^{[1]}(t,w)$, $1/p \to y(t,1/p)$, $1/p \to y^{[1]}(t,1/p)$ from $L^1(K,\mathbb{C})$ to $\mathbb{C}$ as well as the maps $\lambda \to y(t,\lambda)$, $\lambda \to y^{[1]}(t,\lambda)$ from $\mathbb{C}$ to $\mathbb{C}$ are Fréchet differentiable and their derivatives are given by*

$$y'(t,q)(r) = -\int_c^t \phi_{1,2}(t,s)y(t,s)r(s)ds,\ r \in L^1(K,\mathbb{C}),$$

$$(py')'(t,q)(r) = -\int_c^t \phi_{2,2}(t,s)y(t,s)r(s)ds,\ r \in L^1(K,\mathbb{C}),$$

$$y'(t,1/p)(r) = \int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s (py')(x)r(x)dx\right)ds + \int_c^t (py')(x)r(x)dx,\ r \in L^1(K,\mathbb{C}),$$

$$(py')'(t,1/p)(r) = \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (py')(x)r(x)dx\right)ds,\ r \in L^1(K,\mathbb{C}),$$

$$y'(t,w)(r) = \lambda \int_c^t \phi_{1,2}(t,s,w)y(s,w)r(s)ds,\ r \in L^1(K,\mathbb{C}),$$

$$(py')'(t,w)(r) = \lambda \int_c^t \phi_{2,2}(t,s,w)y(s,w)r(s)ds,\ r \in L^1(K,\mathbb{C}),$$

$$y'(t,\lambda)(r) = \int_c^t \phi_{1,2}(t,s,\lambda)w(s)y(s,\lambda)ds,\ \lambda \in \mathbb{C},$$

$$(py')'(t,\lambda)(r) = \int_c^t \phi_{2,2}(t,s,\lambda)w(s)y(s,\lambda)ds,\ \lambda \in \mathbb{C}.$$

Proof. (1) We will prove the first formula.

Let

$$-(py')' + qy = 0,\ y(c) = h,\ (py')'(c) = k;$$
$$-(pz')' + (q+r)y = 0,\ z(c) = h,\ (pz')'(c) = k.$$

Let

$$x = z - y = y(t,q+r) - y(t,q).$$

Then we have

$$-(px')' + qx = -rz,\ x(c) = 0,\ (px')(c) = 0.$$

Let us define the matrix X to be

$$X = \begin{bmatrix} x \\ px' \end{bmatrix},$$

the matrix F to be

$$F = \begin{bmatrix} 0 \\ -rz \end{bmatrix},$$

and the matrix Φ to be

$$\Phi = \begin{bmatrix} \phi_{1,1} & \phi_{2,1} \\ \phi_{2,1} & \phi_{2,2} \end{bmatrix}.$$

From the variation of parameters formula it follows that

$$\begin{aligned} X(t) &= \int_u^t \Phi(t,s)F(s)ds = \int_u^t \Phi(t,s)F(s)ds & (3.12)\\ &= \int_u^t \begin{bmatrix} \phi_{1,1}(t,s) & \phi_{2,1}(t,s) \\ \phi_{2,1}(t,s) & \phi_{2,2}(t,s) \end{bmatrix} \begin{bmatrix} 0 \\ (-r(t))z(t) \end{bmatrix} ds & (3.13)\\ &= \int_u^t \begin{bmatrix} \phi_{1,2}(-r(t))z(t) \\ \phi_{2,2}(-r(t))z(t) \end{bmatrix} ds. & (3.14) \end{aligned}$$

Hence, from equation (3.14), we know that

$$\begin{aligned} x(t) &= \int_c^t \phi_{1,2}(t,s)(-r(s))z(s)ds & (3.15)\\ px'(t) &= \int_c^t \phi_{2,2}(t,s)(-r(s))z(s)ds. & (3.16) \end{aligned}$$

For equation (3.15), letting $z = y + (z-y)$ we get

$$\begin{aligned} z(t)-y(t) &= \int_c^t \phi_{1,2}(t,s)(-r(s))[y(s)+(z(s)-y(s))]ds \\ &= \int_c^t \phi_{1,2}(t,s)(-r(s))y(s)ds + \int_c^t \phi_{1,2}(t,s)[z(s)-y(s)](-r(s))ds \\ &= -\int_c^t \phi_{1,2}(t,s)r(s)y(s)ds - \int_c^t \phi_{1,2}(t,s)[z(s)-y(s)]r(s)ds \end{aligned}$$

so that

$$\begin{aligned} & z(t)-y(t) + \int_c^t \phi_{1,2}(t,s)(r(s))y(s)ds \\ = & -\int_c^t \phi_{1,2}(t,s)[z(s)-y(s)]r(s)ds. \end{aligned}$$

Note that $\phi_{1,2}$ is bounded on $K \times K$ and $z \to y$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{1,2}| \leq M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s) - y(s) = \frac{\epsilon}{M(t-c)}$$

as $r \to 0$ in $L^1(J, \mathbb{C})$. Thus,

$$\begin{aligned}
&\frac{\| - \int_c^t \phi_{1,2}(t,s)[z(s)-y(s)]r(s)ds\|}{\|r\|} \\
\leq\ & \frac{\|\|r\| \int_c^t \phi_{1,2}(t,s)[z(s)-y(s)]ds\|}{\|r\|} \leq \left\| \int_c^t \phi_{1,2}(t,s)[z(s)-y(s)]ds \right\| \\
\leq\ & M \left\| \int_c^t [z(s)-y(s)]ds \right\| \\
=\ & M \left\| \int_c^t \frac{\epsilon}{M(t-c)} ds \right\| \\
=\ & M \frac{\epsilon}{M(t-c)} (t-c) = \epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}
z(t) - y(t) + \int_c^t \phi_{1,2}(t,s)(r(s))y(s)ds &= -\int_c^t \phi_{1,2}(t,s)[z(s)-y(s)]r(s)ds \\
&= o(r) \text{ as } r \to 0 \text{ in } L^1(J, \mathbb{C}).
\end{aligned}$$

Note that

$$\lim_{r\to 0} \frac{\|y(t,q+r) - y(t,q) - T(t,q)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = \frac{\|z(t) - y(t) - [D_2(y)](t,q)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.$$

From the definition of Fréchet derivative, we know that

$$[D_2(y)](t,q)(r) = -\int_c^t \phi_{1,2}(t,s)y(t,s)r(s)ds$$

is indeed the derivative of $y(t,q)$ with respect to q.

(2) For equation (3.16), letting $z = y + (z - y)$ we get

$$\begin{aligned}
pz'(t) - py'(t) &= \int_c^t \phi_{2,2}(t,s)(-r(s))[y(s) + (z(s)-y(s))]ds \\
&= \int_c^t \phi_{2,2}(t,s)(-r(s))y(s)ds + \int_c^t \phi_{2,2}(t,s)[z(s)-y(s)](-r(s))ds \\
&= -\int_c^t \phi_{2,2}(t,s)r(s)y(s)ds - \int_c^t \phi_{2,2}(t,s)[z(s)-y(s)]r(s)ds
\end{aligned}$$

so that

$$z(t) - y(t) + \int_c^t \phi_{2,2}(t,s)(r(s))y(s)ds = -\int_c^t \phi_{2,2}(t,s)[z(s) - y(s)]r(s)ds.$$

Note that $\phi_{2,2}$ is bounded on $K \times K$ and $z \to y$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{2,2}| \leq M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s) - y(s) = \frac{\epsilon}{M(t-c)}$$

as $r \to 0$ in $L^1(J, \mathbb{C})$. Thus, following the same steps as in the last part, we have

$$\begin{aligned} & \frac{\| - \int_c^t \phi_{2,2}(t,s)[z(s) - y(s)]r(s)ds\|}{\|r\|} \\ \leq \;& \frac{\|\|r\| \int_c^t \phi_{2,2}(t,s)[z(s) - y(s)]ds\|}{\|r\|} \\ \leq \;& M \left\| \int_c^t [z(s) - y(s)]ds \right\| \\ = \;& M \frac{\epsilon}{M(t-c)} (t-c) \\ = \;& \epsilon, \end{aligned}$$

which implies that

$$\begin{aligned} pz'(t) - py'(t) + \int_c^t \phi_{2,2}(t,s)(r(s))y(s)ds &= -\int_c^t \phi_{2,2}(t,s)[z(s) - y(s)]r(s)ds \\ &= o(r) \text{ as } r \to 0 \text{ in } L^1(J, \mathbb{C}). \end{aligned}$$

Note that

$$\lim_{r \to 0} \frac{\|py'(t, q + r) - py'(t, q) - T(t, q)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = \frac{\|pz(t) - py(t) - [D_2(py')](t, q)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.$$

From the definition of Fréchet derivative, we know that

$$[D_2(py')](t, q)(r) = -\int_c^t \phi_{2,2}(t,s)y(t,s)r(s)ds$$

is indeed the derivative of $py'(t, q)$ with respect to q.

(3) In order to derive the formulas for the derivatives with respect to $1/p$ we proceed as follows.

Let

$$\frac{1}{p_r} = \frac{1}{p} + r,\ r \in L^1(J, \mathbb{C})$$

and let $y = y(t, 1/p)$, $z = z(t, 1/p_r)$. Set

$$x(t) = \int_c^t \frac{1}{p}(py' - p_r z')ds.$$

Then, we have

$$-(px')' + qx = f,\ x(c) = 0,\ (px')(c) = 0,\ f(t) = -q(t)\int_c^t (p_r z')r ds.$$

Let us define the matrix X to be

$$X = \begin{bmatrix} x \\ px' \end{bmatrix},$$

the matrix F to be

$$F = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

and the matrix Φ to be

$$\Phi = \begin{bmatrix} \phi_{1,1} & \phi_{2,1} \\ \phi_{2,1} & \phi_{2,2} \end{bmatrix}.$$

From the variation of parameters formula it follows that

$$\begin{aligned} X(t) &= \int_u^t \Phi(t,s)F(s)ds && (3.17) \\ &= \int_u^t \Phi(t,s)F(s)ds && (3.18) \\ &= \int_u^t \begin{bmatrix} \phi_{1,1}(t,s) & \phi_{2,1}(t,s) \\ \phi_{2,1}(t,s) & \phi_{2,2}(t,s) \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} ds && (3.19) \\ &= \int_u^t \begin{bmatrix} \phi_{1,2} f(t) \\ \phi_{2,2} f(t) \end{bmatrix} ds && (3.20) \end{aligned}$$

Hence, from equation (3.20), we know that

$$x(t) = \int_c^t \phi_{1,2}(t,s)f(s)ds = -\int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s (p_r z')(u)r(u)du\right)ds \quad (3.21)$$

$$px'(t) = \int_c^t \phi_{2,2}(t,s)f(s)ds = -\int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (p_r z')(u)r(u)du\right)ds. \quad (3.22)$$

For equation (3.21), note that

$$\begin{aligned} z(t)-y(t) &= \int_c^t \left[\frac{1}{p_r}(p_r z') - \frac{1}{p}(py')\right]ds \\ &= \int_c^t \left[\left(\frac{1}{p}+r\right)(p_r z') - \frac{1}{p}(py')\right]ds \\ &= \int_c^t \left[\frac{1}{p}(p_r z') - \frac{1}{p}(py')\right] + \int_c^t (p_r z')rds \\ &= -x(t) + \int_c^t (p_r z')rds \\ &= \int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s (p_r z')(u)r(u)du\right)ds + \int_c^t (p_r z')rds. \end{aligned}$$

Letting $p_r z' = py' + [p_r z' - py']$, we get

$$\begin{aligned} & \int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s (p_r z')(u)r(u)du\right)ds + \int_c^t (p_r z')rds \\ = & \int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s (py')(u)r(u)du\right)ds + \int_c^t (py')rds \\ & + \int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)r(u)du\right)ds + \int_c^t [p_r z' - py']rds. \end{aligned}$$

Hence, we know that

$$\begin{aligned} & z(t)-y(t) - \left(\int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s (py')(u)r(u)du\right)ds + \int_c^t (py')rds\right) \\ = & \int_c^t \phi_{1,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)r(u)du\right)ds + \int_c^t [p_r z' - py']rds. \end{aligned}$$

Note that $\phi_{1,2}$ is bounded on $K \times K$, $q \in L^1(K, \mathbb{C})$ and $p_r z' \to py'$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that

$|\phi_{1,2}q+1| \leq M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s) - y(s) = \frac{\epsilon}{M(t-c)^2}$$

as $r \to 0$ in $L^1(J, \mathbb{C})$. Thus,

$$\begin{aligned}
& \frac{\| \int_c^t \phi_{1,2}(t,s)q(s) \left(\int_c^s [p_r z' - py'](u)r(u)du \right) ds + \int_c^t [p_r z' - py']r \|}{\|r\|} \\
\leq & \frac{\|r\| \| \int_c^t \phi_{1,2}(t,s)q(s) \left(\int_c^s [p_r z' - py'](u)du \right) ds + \int_c^t [p_r z' - py'] \|}{\|r\|} \\
\leq & \left\| \int_c^t \phi_{1,2}(t,s)q(s) \left(\int_c^s [p_r z' - py'](u)du \right) ds + \int_c^t [p_r z' - py'] \right\| \\
= & \left\| \int_c^t (\phi_{1,2}(t,s)q(s) + 1) \left(\int_c^s [p_r z' - py'](u)du \right) ds \right\| \\
= & M \left\| \int_c^t \int_c^t \frac{\epsilon}{M(t-c)^2} du\, ds \right\| \\
= & M \frac{\epsilon}{M(t-c)^2} (t-c)^2 = \epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}
& z(t) - y(t) - \left(\int_c^t \phi_{1,2}(t,s)q(s) \left(\int_c^s (py')(u)r(u)du \right) ds + \int_c^t (py')r \right) \\
= \; & o(r) \text{ as } r \to 0 \text{ in } L^1(J, \mathbb{C}).
\end{aligned}$$

Note that

$$\begin{aligned}
& \lim_{r \to 0} \frac{\|y(t, 1/p + r) - y(t, 1/p) - T(t, 1/p)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} \\
= \; & \frac{\|z(t) - y(t) - [D_2(y)](t, 1/p)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.
\end{aligned}$$

From the definition of Fréchet derivative, we know that

$$[D_2(y)](t, 1/p)(r) = \int_c^t \phi_{1,2}(t,s)q(s) \left(\int_c^s (py')(x)r(x)dx \right) ds + \int_c^t (py')(x)r(x)dx$$

is indeed the derivative of $y(t, 1/p)$ with respect to $1/p$.

(4) Now, we want to derive the Fréchet derivative of $py'(t, 1/p)$ with respect to $1/p$.

Note that
$$px' = p\left(\frac{1}{p}(py' - p_r z')\right) = py' - p_r z'.$$
From equation (3.22), we have
$$p_r z' - py' \;=\; -(px') = \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (p_r z')(u)r(u)du\right) ds.$$

Letting $p_r z' = py' + [p_r z' - py']$, we get
$$\begin{aligned}
&\int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (p_r z')(u)r(u)du\right) ds \\
= \;&\int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (py')(u)r(u)du\right) ds + \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)r(u)du\right) ds.
\end{aligned}$$

Hence, we know that
$$\begin{aligned}
p_r z' - py' \;=\; &\int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (py')(u)r(u)du\right) ds \\
+ \;&\int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)r(u)du\right) ds
\end{aligned}$$

Note that $\phi_{2,2}$ is bounded on $K \times K$, $q \in L^1(K,\mathbb{C})$ and $p_r z' \to py'$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{2,2}q| \leq M$ for some positive real number M. Let $\epsilon > 0$ be given and let
$$z(s) - y(s) = \frac{\epsilon}{M(t-c)^2}$$
as $r \to 0$ in $L^1(J,\mathbb{C})$. Thus,
$$\begin{aligned}
&\frac{\| \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)r(u)du\right) ds\|}{\|r\|} \\
\leq \;&\frac{\|r\|\| \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)du\right) ds\|}{\|r\|} \\
\leq \;&\left\|\int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s [p_r z' - py'](u)du\right) ds\right\| \\
= \;&M\left\|\int_c^t \int_c^t \frac{\epsilon}{M(t-c)^2}\, du\, ds\right\| = M\,\frac{\epsilon}{M(t-c)^2}\,(t-c)^2 = \epsilon,
\end{aligned}$$

which implies that

$$
\begin{aligned}
& (p_r z')(t) - (py')(t) - \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (py')(u)r(u)du\right)ds \\
= \ & o(r) \text{ as } r \to 0 \text{ in } L^1(J,\mathbb{C}).
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \lim_{r\to 0} \frac{\|(py')(t,1/p+r) - (py')(t,1/p) - T(t,1/p)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} \\
= \ & \frac{\|(p_r z')(t) - (py')(t) - [D_2(py')](t,1/p)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.
\end{aligned}
$$

From the definition of Fréchet derivative, we know that

$$
[D_2(py')](t,1/p)(r) = \int_c^t \phi_{2,2}(t,s)q(s)\left(\int_c^s (py')(x)r(x)dx\right)ds
$$

is indeed the derivative of $(py')(t,1/p)$ with respect to $1/p$.

(5) We will prove the next two formulas. Let

$$
\begin{aligned}
-(py')' + qy = \lambda wy,\ y(c) = h,\ (py')'(c) = k; \\
-(pz')' + qy = \lambda(w+r)z,\ z(c) = h,\ (pz')'(c) = k.
\end{aligned}
$$

Let $x = z - y = y(t, w+r) - y(t,w)$. Then

$$
-(px')' + (q - \lambda w)x = \lambda r(t)z,\ x(c) = 0,\ (px')(c) = 0.
$$

Let us define the matrix X to be

$$
X = \begin{bmatrix} x \\ px' \end{bmatrix},
$$

the matrix F to be

$$
F = \begin{bmatrix} 0 \\ \lambda r(t)z \end{bmatrix},
$$

and the matrix Φ to be

$$
\Phi = \begin{bmatrix} \phi_{1,1} & \phi_{2,1} \\ \phi_{2,1} & \phi_{2,2} \end{bmatrix}.
$$

From the variation of parameters formula it follows that

$$\begin{aligned}
X(t) &= \int_u^t \Phi(t,s)F(s)ds = \int_u^t \Phi(t,s)F(s)ds &(3.23)\\
&= \int_u^t \begin{bmatrix} \phi_{1,1}(t,s,w) & \phi_{2,1}(t,s,w) \\ \phi_{2,1}(t,s,w) & \phi_{2,2}(t,s,w) \end{bmatrix} \begin{bmatrix} 0 \\ \lambda r(t)z(t,w) \end{bmatrix} ds &(3.24)\\
&= \int_u^t \begin{bmatrix} \phi_{1,2}\lambda r(t)z(t,w) \\ \phi_{2,2}\lambda r(t)z(t,w) \end{bmatrix} ds &(3.25)
\end{aligned}$$

Hence, from equation (3.25), we know that

$$\begin{aligned}
x(t) &= \lambda \int_c^t \phi_{1,2}(t,s,w)r(s)z(s,w)ds &(3.26)\\
px'(t) &= \lambda \int_c^t \phi_{2,2}(t,s,w)r(s)z(s,w)ds. &(3.27)
\end{aligned}$$

For equation (3.26), letting $z = y + (z - y)$ we get

$$\begin{aligned}
z(t,w) - y(t,w) &= \lambda \int_c^t \phi_{1,2}(t,s,w)r(s)[y(s,\lambda) + (z(s,w) - y(s,w))]ds \\
&= \lambda \int_c^t \phi_{1,2}(t,s,w)r(s)y(s,w)ds \\
&+ \lambda \int_c^t \phi_{1,2}(t,s,w)[z(s,w) - y(s,w)]r(s)ds.
\end{aligned}$$

so that

$$\begin{aligned}
&z(t,w) - y(t,w) - \lambda \int_c^t \phi_{1,2}(t,s,w)r(s)y(s,w)ds \\
&= \lambda \int_c^t \phi_{1,2}(t,s,w)[z(s,w) - y(s,w)]r(s)ds.
\end{aligned}$$

Note that $\phi_{1,2}$ is bounded on $K \times K$, $\lambda \in \mathbb{C}$, and $z \to y$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{1,2}| \le M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s,\lambda) - y(t,w) = \frac{\epsilon}{\lambda M(t-c)}$$

as $r \to 0$ in $\mathbb{C}$. Thus,

$$\begin{aligned}
& \frac{\|\lambda \int_c^t \phi_{1,2}(t,s,w)[z(s,\lambda)-y(s,\lambda)]r(s)ds\|}{\|r\|} \\
\leq\ & \frac{\|\|r\|\lambda \int_c^t \phi_{1,2}(t,s,w)[z(s,\lambda)-y(s,\lambda)]ds\|}{\|r\|} \\
\leq\ & \left\|\lambda \int_c^t \phi_{1,2}(t,s,w)[z(s,\lambda)-y(s,\lambda)]ds\right\| \\
\leq\ & M\left\|\lambda \int_c^t [z(s,\lambda)-y(s,\lambda)]ds\right\| = M\left\|\lambda \int_c^t \frac{\epsilon}{\lambda M(t-c)}ds\right\| \\
=\ & M\,\lambda\,\frac{\epsilon}{\lambda M(t-c)}\,(t-c) = \epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}
& z(t,w)-y(t,w)-\lambda\int_c^t \phi_{1,2}(t,s,w)r(s)y(s,w)ds \\
=\ & \lambda\int_c^t \phi_{1,2}(t,s,w)[z(s,w)-y(s,w)]r(s)ds = o(r) \text{ as } r\to 0 \text{ in } \mathbb{C}.
\end{aligned}$$

Note that

$$\lim_{r\to 0}\frac{\|y(t,w+r)-y(t,w)-T(t,w)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = \frac{\|z(t,w)-y(t,w)-[D_2(y)](t,w)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.$$

From the definition of Fréchet derivative, we know that

$$[D_2(y)](t,w) = \lambda\int_c^t \phi_{1,2}(t,s,w)y(s,w)r(s)ds$$

is indeed the derivative of $y(t,w)$ with respect to w.

(6) For equation (3.27), letting $z = y + (z-y)$ we get

$$\begin{aligned}
(pz')(t,w)-(py')(t,w) &= \lambda\int_c^t \phi_{2,2}(t,s,w)r(s)[y(s,w)+(z(s,w)-y(s,w))]ds \\
&= \lambda\int_c^t \phi_{2,2}(t,s,w)r(s)y(s,w)ds \\
&+ \lambda\int_c^t \phi_{2,2}(t,s,w)[z(s,w)-y(s,w)]r(s)ds.
\end{aligned}$$

so that

$$\begin{aligned}
& (pz')(t,w) - (py')(t,w) - \lambda \int_c^t \phi_{2,2}(t,s,w) r(s) y(s,w) ds \\
= \quad & \lambda \int_c^t \phi_{2,2}(t,s,w)[z(s,w) - y(s,w)] r(s) ds.
\end{aligned}$$

Note that $\phi_{2,2}$ is bounded on $K \times K$, $\lambda \in \mathbb{C}$, and $z \to y$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{2,2}| \leq M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s,\lambda) - y(t,w) = \frac{\epsilon}{\lambda M(t-c)}$$

as $r \to 0$ in $\mathbb{C}$. Thus,

$$\begin{aligned}
& \frac{\|\lambda \int_c^t \phi_{2,2}(t,s,w)[z(s,\lambda) - y(s,\lambda)] r(s) ds\|}{\|r\|} \\
\leq \quad & \frac{\|\|r\| \lambda \int_c^t \phi_{2,2}(t,s,w)[z(s,\lambda) - y(s,\lambda)] w(s) ds\|}{\|r\|} \\
\leq \quad & \left\| \lambda \int_c^t \phi_{2,2}(t,s,w)[z(s,\lambda) - y(s,\lambda)] ds \right\| \\
\leq \quad & M \left\| \lambda \int_c^t [z(s,\lambda) - y(s,\lambda)] ds \right\| = M \left\| \lambda \int_c^t \frac{\epsilon}{\lambda M(t-c)} ds \right\| \\
= \quad & M \, \lambda \, \frac{\epsilon}{\lambda M(t-c)} \, (t-c) = \epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}
& (pz')(t,w) - (py')(t,w) - \lambda \int_c^t \phi_{2,2}(t,s,w) r(s) y(s,w) ds \\
= \quad & \lambda \int_c^t \phi_{2,2}(t,s,w)[z(s,w) - y(s,w)] r(s) ds \\
= \quad & o(r) \text{ as } r \to 0 \text{ in } \mathbb{C}.
\end{aligned}$$

Note that

$$\begin{aligned}
& \lim_{r \to 0} \frac{\|(py')(t,w+r) - (py')(t,w) - T(t,w)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} \\
= \quad & \frac{\|z(t,w) - y(t,w) - [D_2(py')](t,w)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.
\end{aligned}$$

From the definition of Fréchet derivative, we know that

$$[D_2(py')](t,w) = \lambda \int_c^t \phi_{2,2}(t,s,w)y(s,w)r(s)ds$$

is indeed the derivative of $(py')(t,w)$ with respect to w.

(7) We will prove the last two formulas.

Let

$$-(py')' + qy = \lambda wy,\ y(c) = h,\ (py')'(c) = k;$$
$$-(pz')' + qy = (\lambda + r)wz,\ z(c) = h,\ (pz')'(c) = k.$$

Let

$$x = z - y = y(t, \lambda + r) - y(t, \lambda).$$

Then

$$-(px')' + (q - \lambda w)x = rwz,\ x(c) = 0,\ (px')(c) = 0.$$

Let us define the matrix X to be

$$X = \begin{bmatrix} x \\ px' \end{bmatrix},$$

the matrix F to be

$$F = \begin{bmatrix} 0 \\ rwz \end{bmatrix},$$

and the matrix Φ to be

$$\Phi = \begin{bmatrix} \phi_{1,1} & \phi_{2,1} \\ \phi_{2,1} & \phi_{2,2} \end{bmatrix}.$$

From the variation of parameters formula it follows that

$$\begin{aligned} X(t) &= \int_u^t \Phi(t,s)F(s)ds = \int_u^t \Phi(t,s)F(s)ds && (3.28)\\ &= \int_u^t \begin{bmatrix} \phi_{1,1}(t,s,\lambda) & \phi_{2,1}(t,s,\lambda) \\ \phi_{2,1}(t,s,\lambda) & \phi_{2,2}(t,s,\lambda) \end{bmatrix} \begin{bmatrix} 0 \\ rw(t)z(t,\lambda) \end{bmatrix} ds && (3.29)\\ &= \int_u^t \begin{bmatrix} \phi_{1,2} rw(t)z(t,\lambda) \\ \phi_{2,2} rw(t)z(t,\lambda) \end{bmatrix} ds && (3.30) \end{aligned}$$

Hence, from equation (3.30), we know that

$$x(t) = \int_c^t \phi_{1,2}(t,s,\lambda) r w(s) z(s,\lambda) ds \tag{3.31}$$

$$px'(t) = \int_c^t \phi_{2,2}(t,s,\lambda) r w(s) z(s,\lambda) ds. \tag{3.32}$$

For equation (3.31), letting $z = y + (z - y)$ we get

$$\begin{aligned} z(t,\lambda) - y(t,\lambda) &= \int_c^t \phi_{1,2}(t,s,\lambda) r w(s)[y(s,\lambda) + (z(s,\lambda) - y(s,\lambda))] ds \\ &= \int_c^t \phi_{1,2}(t,s,\lambda) r w(s) y(s,\lambda) ds \\ &+ \int_c^t \phi_{1,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] r w(s) ds. \end{aligned}$$

so that

$$z(t,\lambda) - y(t,\lambda) - \int_c^t \phi_{1,2}(t,s,\lambda) r w(s) y(s,\lambda) ds = \int_c^t \phi_{1,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] r w(s) ds.$$

Note that $\phi_{1,2}$ is bounded on $K \times K$, $w \in L^1_{loc}(J, \mathbb{C})$, and $z \to y$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{1,2} w| \leq M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s,\lambda) - y(s,\lambda) = \frac{\epsilon}{M(t-c)}$$

as $r \to 0$ in $\mathbb{C}$. Thus,

$$\begin{aligned} &\frac{\| \int_c^t \phi_{1,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] r w(s) ds \|}{\|r\|} \\ \leq\; &\frac{\| \|r\| \int_c^t \phi_{1,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] w(s) ds \|}{\|r\|} \\ \leq\; &\left\| \int_c^t \phi_{1,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] w(s) ds \right\| \\ \leq\; &M \left\| \int_c^t [z(s,\lambda) - y(s,\lambda)] ds \right\| \\ =\; &M \left\| \int_c^t \frac{\epsilon}{M(t-c)} ds \right\| = M \frac{\epsilon}{M(t-c)} (t-c) = \epsilon, \end{aligned}$$

which implies that

$$\begin{aligned} & z(t,\lambda) - y(t,\lambda) - \int_c^t \phi_{1,2}(t,s,\lambda) r w(s) y(s,\lambda) ds \\ = & \int_c^t \phi_{1,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] r w(s) ds = o(r) \text{ as } r \to 0 \text{ in } \mathbb{C}. \end{aligned}$$

Note that

$$\lim_{r\to 0} \frac{\|y(t,\lambda+r) - y(t,\lambda) - T(t,\lambda)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = \frac{\|z(t,\lambda) - y(t,\lambda) - [D_2(y)](t,\lambda)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.$$

From the definition of Fréchet derivative, we know that

$$[D_2(y)](t,\lambda) = \int_c^t \phi_{1,2}(t,s,\lambda) y(s,\lambda) w(s) ds$$

is indeed the derivative of $y(t,\lambda)$ with respect to λ.

(8) For equation (3.32), letting $z = y + (z - y)$ we get

$$\begin{aligned} (pz')(t,\lambda) - (py')(t,\lambda) &= \int_c^t \phi_{2,2}(t,s,\lambda) r w(s)[y(s,\lambda) + (z(s,\lambda) - y(s,\lambda))] ds \\ &= \int_c^t \phi_{2,2}(t,s,\lambda) r w(s) y(s,\lambda) ds \\ &+ \int_c^t \phi_{2,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] r w(s) ds. \end{aligned}$$

so that

$$\begin{aligned} & (pz')(t,\lambda) - (py')(t,\lambda) - \int_c^t \phi_{2,2}(t,s,\lambda) r w(s) y(s,\lambda) ds \\ = & \int_c^t \phi_{2,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)] r w(s) ds. \end{aligned}$$

Note that $\phi_{2,2}$ is bounded on $K \times K$, $w \in L^1_{loc}(J,\mathbb{C})$, and $z \to y$ uniformly on K by the continuous dependence of solutions in a compact interval. Suppose that $|\phi_{2,2} w| \le M$ for some positive real number M. Let $\epsilon > 0$ be given and let

$$z(s,\lambda) - y(s,\lambda) = \frac{\epsilon}{M(t-c)}$$

as $r \to 0$ in $\mathbb{C}$. Thus,

$$\begin{aligned}
& \frac{\| \int_c^t \phi_{2,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)]rw(s)ds\|}{\|r\|} \\
\leq\ & \frac{\|\|r\| \int_c^t \phi_{2,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)]w(s)ds\|}{\|r\|} \\
\leq\ & \left\| \int_c^t \phi_{2,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)]w(s)ds \right\| \\
\leq\ & M \left\| \int_c^t [z(s,\lambda) - y(s,\lambda)]ds \right\| \\
=\ & M \left\| \int_c^t \frac{\epsilon}{M(t-c)} ds \right\| \\
=\ & M \frac{\epsilon}{M(t-c)} (t-c) \\
=\ & \epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}
& (pz')(t,\lambda) - (py')(t,\lambda) - \int_c^t \phi_{2,2}(t,s,\lambda)rw(s)y(s,\lambda)ds \\
=\ & \int_c^t \phi_{2,2}(t,s,\lambda)[z(s,\lambda) - y(s,\lambda)]rw(s)ds \\
=\ & o(r) \text{ as } r \to 0 \text{ in } \mathbb{C}.
\end{aligned}$$

Note that

$$\begin{aligned}
& \lim_{r\to 0} \frac{\|(py')(t,\lambda + r) - (py')(t,\lambda) - T(t,\lambda)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} \\
=\ & \frac{\|z(t,\lambda) - y(t,\lambda) - [D_2(py')](t,\lambda)(r)\|_{\mathbb{C}}}{\|r\|_{\mathbb{C}}} = 0.
\end{aligned}$$

From the definition of Fréchet derivative, we know that

$$[D_2(py')](t,\lambda) = \int_c^t \phi_{2,2}(t,s,\lambda)y(s,\lambda)w(s)ds$$

is indeed the derivative of $(py')(t,\lambda)$ with respect to λ. $\square$

3.3 The Prüfer Transformations

3.3.1 Definition

Recall that we have been focusing on second-order differential equations of the form

$$-(py')' + qy = \lambda wy,$$

where

$$J = (a, b), \quad -\infty \leq a < b \leq +\infty, \quad \frac{1}{p},\, q,\, w : L^1_{loc}(J, \mathbb{C}).$$

In this section, we will consider the substitution $y = \rho \sin\theta$ and $py' = \rho\cos\theta$, where

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}.$$

We claim that the differential equation above is equivalent to

$$\theta_j' = r_j \cos^2\theta_j + g_j \sin^2\theta_j$$

on $J = [a, b]$, where $\infty \leq a < b \leq \infty$ with $j = 1, 2$. The initial conditions are

$$\theta_j(c_j) = \alpha_j, \quad c_j \in J, \quad \alpha_j \in \mathbb{R}, \quad j = 1, 2.$$

Moreover, we have

$$r_j = 1/p_j, \quad g_j = \lambda w - q \in L^1(J, \mathbb{R}).$$

Recall that ρ, θ, y, py' are all functions of t. Let $x = py'$, and we have

$$\rho^2 = x^2 + y^2.$$

Taking the derivative of both sides with respect to t, we have

$$2\rho \cdot \rho' = 2x \cdot x' + 2y \cdot y',$$

which implies that

$$\rho = \frac{x \cdot x' + y \cdot y'}{\rho'}.$$

Now, let us consider x' and y'.

$$x' = \rho' \cos\theta - \rho\theta' \sin\theta = \frac{\rho'}{\rho} x - y\theta' \tag{3.33}$$

$$y' = \rho' \sin\theta + \rho\theta' \cos\theta = \frac{\rho'}{\rho} y + x\theta'. \tag{3.34}$$

Multiplying (3.33) by y and (3.34) by x and taking the difference, we have

$$xy' - yx' = (x^2 + y^2)\theta' = \rho^2\theta'.$$

Hence, we have

$$\theta' = \frac{xy' - yx'}{\rho^2}.$$

By our original equation and the substitution,

$$\begin{aligned} x &= \rho\cos\theta, & y' &= \frac{x}{p}, \\ y &= \rho\sin\theta, & x' &= qy - \lambda wy = (q - \lambda w)y. \end{aligned}$$

Thus, we have

$$\begin{aligned} \theta' &= \frac{xy' - yx'}{\rho^2} \\ &= \frac{\rho\cos\theta\frac{x}{p} - \rho\sin\theta(q - \lambda w)y}{\rho^2} \\ &= \frac{\rho\cos\theta\frac{\rho\cos\theta}{p} - \rho\sin\theta(q - \lambda w)\rho\sin\theta}{\rho^2} \\ &= \cos\theta\frac{\cos\theta}{p} - \sin\theta(q - \lambda w)\sin\theta \\ &= \frac{1}{p}\cos^2\theta + (\lambda w - q)\sin^2\theta \\ &= r\cos^2\theta + g\sin^2\theta, \end{aligned}$$

where $r = 1/p$, $\quad g = \lambda w - q$.

Now, we want to construct specific examples to show that the Prüfer transformation is valid.

3.3.2 Examples

Example 3.13. *Let us consider the equation*

$$-(py')' + qy = \lambda wy$$

on $J = (a, b)$*, where*

$$J = (a, b) = (0, 1);\ \frac{1}{p(t)} = t^{-2} \sin\left(\frac{1}{t}\right), \textit{where } 0 < t < 1;\ q = 0;\ w = 0.$$

Then, we have

$$-(py')' = 0$$

As solved above, the possible solutions are

$$u(t) = 1, \quad v(t) = \cos\left(\frac{1}{t}\right).$$

Now, we want to check if the differential equation

$$\theta' = r\cos^2\theta + g\sin^2\theta$$

gives the same results. By Mathematica (codes in the appendix), it is easy to show that the solutions above satisfies the differential equation after the Prüfer transformation.

3.4 Oscillation

In this subsection, we will study the oscillatory properties of the equation

$$My = (py')' + qy = 0 \quad \text{on} \quad J = (a, b), \quad -\infty \le a < b \le \infty, \tag{3.35}$$

where the coefficients are assumed to satisfy

$$1/p, q \in L^1_{loc}(J, \mathbb{R}), \quad p > 0 \text{ a.e. on } J. \tag{3.36}$$

We can think of M as an operator on the function y.

3.4.1 Definitions

Definition 3.14. (Principal and Non-principal Solutions). Let u, v be real solutions of (3.35). Then

- u is called a principal solution at a if
  (1) $u(t) \neq 0$ for $t \in (a, d]$ and some $d \in J$,
  (2) every solution y of (3.35) which is not a multiple of u satisfies $u(t)/y(t) \to 0$ as $t \in a^+$.
  Note that $y(t) \neq 0$ for t in some right neighborhood of a by the Sturm Separation Theorem.

- v is called a non-principal solution at a if
  (1) $v(t) \neq 0$ for $t \in (a, d]$ and some $d \in J$,
  (2) v is not a principal solution at a.

Principal and non-principal solutions at b are defined similarly.

Definition 3.15. (Oscillation and Non-Oscillation). Let (3.35) and (3.36) hold. Then endpoint a is called oscillatory (O) if there is a nontrivial solution y which has a zero in the interval (a, c) for every c in J. Similarly b is oscillatory if there is nontrivial solution which has a zero in the interval (c, b) for every c in J. By the Sturm Separation Theorem, if one nontrivial solution has this property at an endpoint then all nontrivial solutions have this property. Hence oscillation is a property of the equation (3.35) and does not depend on any particular solution. An endpoint is non-oscillatory (NO) if ti is not oscillatory.

3.4.2 Oscillation Criteria

In this section, we will discuss various criteria for equation (3.35) to be oscillatory or non-oscillatory at a particular endpoint.

Theorem 3.16. *The equation* (3.35) *is non-oscillatory at a if and only if there exists a principal solution at a.*

Proof. Since by definition a principal solution at a is non-oscillatory at a, we need only show the converse. Let y_1 and y_2 be linearly independent real solutions of (3.35). Since

(3.35) is non-oscillatory, it follows from the Sturm Separation Theorem that there is an α in I such that $y_1(t), y_2(t) \neq 0$ for $a < t < \alpha$. Since y_1, y_2 are linearly independent real solutions, it follows that

$$\lim_{t \to a} \frac{y_1(t)}{y_2(t)} = L, \quad \text{where } -\infty \leq L \leq \infty,$$

exists.

If L is finite, define $u = y_1 - Ly_2, v = y_2$; otherwise, define $u = y_2$, $v = y_1$. In any case, u and v are real linearly independent solutions of (3.35) such that

$$\lim_{t \to a} \frac{u(t)}{v(t)} = 0.$$

Let y be any solution of (3.35) which is not a multiple of u. Then there exist constants c, d such that $d \neq 0$ and $y = cu + dv$. Therefore,

$$\lim_{t \to a} \left\| \frac{y(t)}{u(t)} \right\| = \lim_{t \to a} \left\| c + d\frac{v(t)}{u(t)} \right\| = \infty.$$

Hence, by definition, $u(t) = o(y(t))$ as $t \to a$. This shows that u is a principal solution at a. □

Theorem 3.17. *Let* (3.35), (3.36) *hold.*

- *If*

$$\int_c^b \frac{1}{p} = \infty \quad \textit{and} \quad \int_c^b q = \infty$$

for some $c \in J$, then the equation (3.35) *is oscillatory at b.*

- *If*

$$\int_a^c \frac{1}{p} = \infty \quad \textit{and} \quad \int_a^c q = \infty$$

for some $c \in J$, then the equation (3.35) *is oscillatory at a.*

Proof. We prove the result for the endpoint b only since the proof for a is similar. Assume non-oscillation at b and let u, v be positive principal and non-principal solutions, respectively, on $[d, b)$ for some $d \in J$. Noting that $q = -(pv')'/v$ on $[d, b)$ and integrating by parts we get

$$\begin{aligned} -\int_d^t q &= \int_d^t \frac{(pv')'}{v} = \frac{(pv')}{v}(t) - \frac{(pv')}{v}(d) - \int_d^t pv' \frac{-v'}{v^2} \\ &= \frac{(pv')}{v}(t) - \frac{(pv')}{v}(d) + \int_d^t \frac{1}{p}\frac{(pv')^2}{v^2}. \end{aligned}$$

Note that

$$-\int_d^t q = -\infty$$

as $t \to b$. Hence, $\dfrac{(pv')(t)}{v(t)} \to -\infty$ as $t \to b$ and pv' is negative near b. Therefore, we know that v must be decreasing in a left neighborhood of b and hence has a limit at b, say,

$$v(t) \to L \quad \text{as} \quad t \to b, \quad , 0 \le L < \infty.$$

From this it follows that, for some $h \in [d, b)$, and some $\epsilon > 0$, we have

$$v^2(t) > L^2 + \epsilon, \quad \frac{1}{v^2(t)} > \frac{1}{L^2+\epsilon}, \quad d \le h \le t < b,$$

which implies that

$$\int_h^t \frac{1}{pv^2} \ge \frac{1}{L^2+\epsilon}\int_h^t \frac{1}{p}, \quad d \le h \le t < b.$$

This and the hypothesis imply that

$$\int_h^b \frac{1}{pv^2} = \infty,$$

contradicting the fact that

$$\int_h^b \frac{1}{pv^2} < \infty$$

since v is a non-principal solution at b. □

Theorem 3.18. *Let $J = (1, \infty)$, $p = 1$ on J, $q \in L^1_{loc}([1,\infty), \mathbb{R})$ and assume that q is piecewise continuous on J. Set $q_+(t) = \max\{q(t), 0\}$, $q_- = \max\{-q(t), 0\}$, $t \in J$. Assume that for each $\epsilon > 0$ there is a $\delta > 0$ such that for any subinterval I of J of length δ*

$$\textit{either } \int_I q_+ < \epsilon \textit{ or } \int_I q_- < \epsilon.$$

If

$$-\infty \leq \liminf_{t\to\infty} \int_1^t q < \limsup_{t\to\infty} \int_1^t q \leq \infty,$$

then equation (3.35) *is oscillatory at* ∞.

Example 3.19. *It follows readily from Theorem 3.18 that the equation*

$$y'' + qy = 0$$

is oscillatory at ∞ *when* $q(t) = k\sin(t)$ *or* $q(t) = k\cos(t)$ *for any* $k \in \mathbb{R}$, $k \neq 0$. *More generally, for* $q(t) = t^c k\sin(t)$ *or* $q(t) = t^c k\cos(t)$, *with* $-1 < c \in \mathbb{R}$, $k \neq 0$.

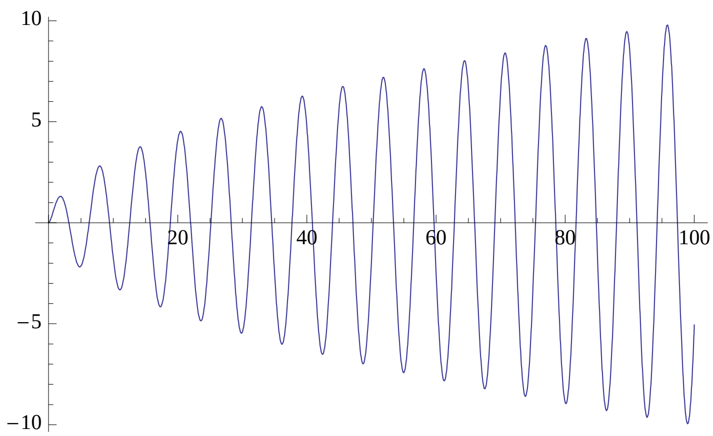

Figure 6: Example 3.19: $q(t) = t^{0.5}\sin(t)$

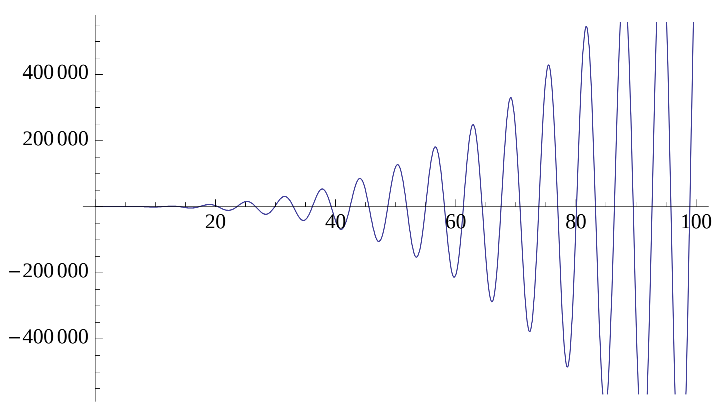

Figure 7: Example 3.19: $q(t) = t^3\cos(t)$.

3.5 Sturm Comparison Theorem

Theorem 3.20. *Consider a singular endpoint of equation*

$$(py')' + qy = 0 \quad on \quad J, \quad 1/p,\, q \in L^1_{loc}(J,\mathbb{R}), \quad p > 0 \ a.e. \quad on \quad J = (a,b). \tag{3.37}$$

Now, consider a comparison equation

$$(Pz')' + Qz = 0 \quad on \quad J. \tag{3.38}$$

Assume $1/P$, $Q \in L^1_{loc}(J,\mathbb{R})$ *and satisfy*

$$Q \geq q, \quad 0 < P \leq p \quad on \quad J. \tag{3.39}$$

Suppose y *is a non-trivial solution of* (3.37) *satisfying* $y(c) = 0 = y(d)$ *for some* $c, d \in J$, $c < d$. *Then every solution of* (3.38) *has a zero in the closed interval* $[c, d]$.

Proof. Since the zeros of y are isolated by Theorem (3.4), we may assume that c and d are consecutive zeros of y. Replacing y by $-y$, if necessary, we may also assume that $y > 0$ on (c, d). Suppose that z is a nontrivial solution of (3.38) which has no zero in $[c, d]$. Then a direct, albeit tedious, computation yields the Picone identity

$$[\frac{y}{z}(py'z - Pz'y)]' = (Q-q)y^2 + (p-P)y'^2 + P\frac{(yz' - zy')^2}{z^2}. \tag{3.40}$$

An integration yields

$$\int_c^d (Q-q)y^2 + \int_c^d (p-P)y'^2 = -\int_c^d P\frac{(yz' - zy')^2}{z^2}. \tag{3.41}$$

The hypothesis (3.39) implies that $yz' - zy' = 0$ a.e. on $[c, d]$. Hence, $f = y/z = 0$, and consequently, $y = 0$ on $[c, d]$. By the existence-uniqueness theorem, this implies that y is the trivial solution on J and this contradiction completes the proof. $\square$

4 Sturm Oscillation and Comparison Theorems

4.1 Basics

4.1.1 Construction of Difference Equations from a Jacobi Matrix

In this case, given the Jacobi matrix

$$J = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & \cdots & \cdots \\ a_1 & b_2 & a_2 & & & \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & \ddots \\ \vdots & & & & \ddots & \ddots \end{bmatrix},$$

we consider the difference equation discussed in the very beginning, that is,

$$(hu)_n = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = xu \tag{4.1}$$

for $n = 1, 2, \cdots$ with $u_0 = 0$.

Given a sequence of parameters $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ for the difference equation (4.1), we look at the fundamental solution, $u_n(x)$, defined recursively by $u_1(x) = 1$ and

$$a_n u_{n+1}(x) + (b_n - x) u_n(x) + a_{n-1} u_{n-1}(x) = 0 \tag{4.2}$$

with $u_0 \equiv 0$. Hence,

$$u_{n+1}(x) = a_n^{-1}(x - b_n) u_n(x) - a_n^{-1} a_{n-1} u_{n-1}(x) = 0 \tag{4.3}$$

Clearly, (4.3) implies, by induction, that u_{n+1} is a polynomial of degree n with leading term $(a_n \cdots a_1)^{-1} x^n$. Thus, we define for $n = 0, 1, 2, \cdots$

$$p_n(x) = u_{n+1}(x), \qquad P_n(x) = (a_1 \cdots a_n) p_n(x). \tag{4.4}$$

Then (4.2) becomes

$$a_{n+1} p_{n+1}(x) + (b_{n+1} - x) p_n(x) + a_n p_{n-1}(x) = 0 \tag{4.5}$$

for $n = 0, 1, 2, \cdots$. One also sees that

$$xP_n(x) = P_{n+1}(x) + b_{n+1}P_n(x) + a_n^2 P_{n-1}(x), \tag{4.6}$$

where p_n are orthonormal polynomials for a suitable measure on $\mathbb{R}$ and the P_n are what are known as monic orthogonal polynomials.

Now, let J_n be the finite $n \times n$ matrix defined by

$$J_n = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & b_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & 0 & a_{n-1} & b_n \end{bmatrix}.$$

Proposition 4.1. *The eigenvalues of J_n are precisely the zeros of $P_n(x)$. We have*

$$P_n(x) = \det(xI - J_n). \tag{4.7}$$

Proof. Let $\varphi(x)$ be the vector where $\varphi_j(x) = p_{j-1}(x)$, $j = 1, \cdots, n$. Then (4.2) implies that

$$(J_n - x)\varphi(x) = -a_n p_n(x)\delta_n, \tag{4.8}$$

where δ_n is the vector $(0, 0, \cdots, 0, 1)$. Thus every zero of p_n is an eigenvalue of J_n. Conversely, if $\tilde{\varphi}$ is an eigenvector of J_n, then both $\tilde{\varphi}_j$ and φ_j solve (4.3), so $\tilde{\varphi}_j = \tilde{\varphi}_1\varphi_j(x)$. This implies that x is an eigenvalue only if $p_n(x)$ is zero and that eigenvalues are simple.

Since J_n is real symmetric and eigenvalues are simple, $p_n(x)$ has n distinct eigenvalues $x_j^{(n)}$, $j = 1, \cdots, n$ with $x_{j-1}^{(n)} < x_j^{(n)}$. Thus, since p_n and P_n have the same zeros, we have

$$P_n(x) = \prod_{i=1}^{n}(x - x_j^n) = \det(x - J_n),$$

which completes the proof. ∎

As proved above, from the Jacobi matrices representation of a polynomial $P_n(x)$, we have

$$P_n(x) = \det(x - J_n). \tag{4.9}$$

The determinant relation (4.9) has an important consequence concerning eigenvalue counting for J_n with carryover to the infinite J. As already noted, the eigenvalues of J_n are exactly the zeros $\{x_j^{(n)}(d\mu)\}_{j=0}^{n-1}$ of $P_n(x)$. Thus, the eigenvalues of J_{n-1} and J_n interlace since that is true of the zeros (this also follows from the fact that J_{n-1} is the restriction of J_n to a codimension 1 subspace.)

4.2 Sturm Oscillation and Comparison Theorems

Theorem 4.2. *(Sturm Oscillation Theorem). If $P_\ell(x_0) \neq 0$, $\ell = 0, 1, \cdots, n$, then the number of eigenvalues of $J_{n;F}$ above x_0 is*

$$\#\{j | 0 \leq j \leq n-1 \textit{ so that } \mathrm{sgn}(P_{j+1}(x_0)) \neq \mathrm{sgn}(P_j(x_0))\}.$$

Proof. See Simon (2005), page 4–5. □

Theorem 4.3. *(Sturm Comparison Theorem) If u and v are linearly independent solutions of*

$$x_0 u_n = a_{n+1} u_{n+1} + b_{n+1} u_n + a_n u_{n-1},$$

v has a sign change between every pair of sign changes of u.

Proof. The proof immediately follows from the proof for the Sturm Comparison Theorem using the differential equations approach. □

5 Connection between the Two Approaches

5.1 Construction of the Jacobi Matrix from Differential Equations

In this section, we will again look at the initial value problems consisting of the second order scalar equation

$$-(py')' + qy = f \text{ on } J \tag{5.1}$$

together with the initial conditions

$$y(c) = 0, \quad y'(c) = 1/2, \quad c \in J, \quad h, k \in \mathbb{C} \tag{5.2}$$

where

$$J = (a, b), \quad -\infty \leq a < b \leq +\infty, \quad \frac{1}{p},\ q,\ f : J \to \mathbb{C}. \tag{5.3}$$

In order to verify that the Sturm-Liouville problem defined using the differential equation is essentially the same as its discrete analog, that is, the difference equations, we need to construct an appropriate Jacobi matrix from the differential equation above, and the difference equations will immediately follow from the Jacobi matrix.

Let $(t_n)_{n=0}^{\infty}$ be a sequence of complex numbers. Let (p_n) be a sequence of points on the function p defined by $p_n = p(t_{n+1})$ for all $n \in \{0, 1, \cdots\}$, (q_n) be a sequence of points on the function q defined by $q_n = q(t_{n+1})$ for all $n \in \{0, 1, \cdots\}$, (f_n) be a sequence of points on the function f defined by $f_n = f(t_{n+1})$ for all $n \in \{0, 1, \cdots\}$. Finally, define a sequence of points y_n on the solution y by $y_n = y(t_{n+1})$. With sufficiently large n, define

$$y_{n+1/2} = \frac{1}{2}(y_{n+1} + y_n).$$

Moreover, define the first derivatives as

$$\begin{aligned} p' &= p_{n+1/2} - p_{n-1/2} \\ y' &= y_{n+1/2} - y_{n-1/2} \end{aligned}$$

and the second derivatives as

$$\begin{aligned} p'' &= p'_{n+1/2} - p'_{n-1/2} = p_{n+1} - 2p_n + p_{n-1} \\ y'' &= y'_{n+1/2} - y'_{n-1/2} = y_{n+1} - 2y_n + y_{n-1}. \end{aligned}$$

Now, we are ready to state a theorem in the special case when $p = 1$ using the definition above:

Theorem 5.1. *For the Schrödinger equation of the form*

$$-y'' + qy = f,$$

where $q, f \in L^1(J, C)$, using the definitions above, we can write its corresponding difference equation as

$$-y_{n+1} + (2 + q_n)y_n - y_{n-1} = f_n.$$

Moreover, its corresponding Jacobi matrix is given by

$$J_n = \begin{bmatrix} 2+q_1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2+q_2 & -1 & & & \vdots \\ 0 & -1 & 2+q_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 2+q_{n-1} & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2+q_n \end{bmatrix}.$$

Proof. The proof immediately follows from the definition. □

The general case is more complicated. Note that by the product rule,

$$(py')' = py'' + p'y'.$$

Hence, we have

$$\begin{aligned} & -(py')' + qy = f \\ \iff & -py'' - p'y' + qy = f \\ \iff & -p_n(y_{n+1} - 2y_n + y_{n-1}) - (p_{n+1/2} - p_{n-1/2})(y_{n+1/2} - y_{n-1/2}) + q_n y_n = f_n \\ \iff & -p_n y_{n+1} + (2p_n + q_n)y_n - p_n y_{n-1} - p_{n+1/2}y_{n+1/2} + p_{n-1/2}y_{n+1/2} \\ & +p_{n+1/2}y_{n-1/2} - p_{n-1/2}y_{n-1/2} = f_n \\ \iff & -((1/2)p_{n+1/2} + p_n - (1/2)p_{n-1/2})y_{n+1} + (2p_n + q_n)y_n \\ & -((1/2)p_{n+1/2} - p_n - (1/2)p_{n-1/2})y_{n-1} = f_n. \end{aligned}$$

In this case, there is no obvious sequence $\{a_n\}$ such that

$$a_{n+1} = -((1/2)p_{n+1/2} + p_n - (1/2)p_{n-1/2})$$

and

$$a_n = -((1/2)p_{n+1/2} - p_n - (1/2)p_{n-1/2}).$$

If such sequence does exist, then letting $b_n = 2p_n + q_n$, and we will have our corresponding Jacobi matrix

$$J_n = \begin{bmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & b_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & \cdots & a_{n-1} & b_n \end{bmatrix}.$$

Now, let $c = t_1$. Note that the initial conditions in this case becomes

$$y(c) = y(t_1) = y_0 = 0$$

and

$$y'(c) = y'(t_1) = y_{3/2} - y_{1/2} = \frac{y_2 + y_1}{2} - \frac{y_1 + y_0}{2} = \frac{y_2 - y_0}{2} \approx \frac{y_1 - y_0}{2} = \frac{y_1}{2} = \frac{1}{2}$$

so that $y_1 = 1$.

5.2 Examples

Example 5.2. *First, we will give an example of Theorem 5.1.*

Let us consider the differential equation

$$y''(x) + y(x) = \sin(x)$$

with boundary conditions $y(0) = 0$ and $y'(0) = 1$. By the Theorem above, we can write its corresponding difference equation as

$$y_{n+2} - y_{n+1} + y_n = \sin(n)$$

with initial conditions $y_0 = 0$ and $y_1 = 1$.

Here, choosing $t_n = n$, we have the following graphs of the solution to the difference equation (the one with smaller amplitude) and that to the differential equation (the one with larger amplitude).

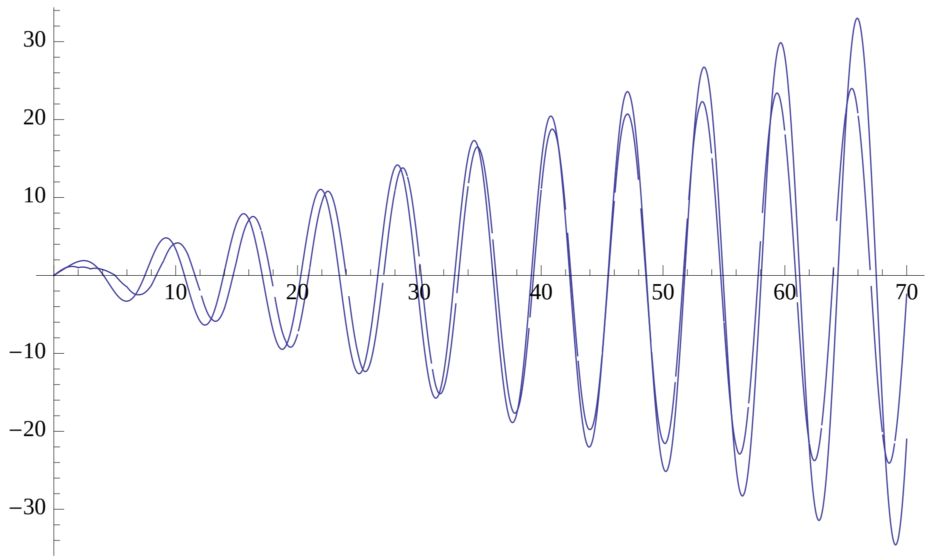

Figure 8: Example 5.2

Zeros of the solution to the differential equation:

{0, 4.0782, 7.4722, 10.7228, 13.925, 17.1051, 20.2734, 23.4346, 26.5912, 29.7446, 32.8958, 36.0453, 39.1935, 42.3408, 45.4872, 48.6331, 51.7784, 54.9233, 58.0678, 61.2121, 64.3561, 67.4998}.

Zeros of the solution to the difference equation:

{0, 3, 5.028, 8.4039, 11.5854, 14.7121, 17.8162, 20, 9089, 23.9948, 27.0712, 30.1465, 33.2213, 36.2959, 39.3703, 42.4445, 45.5185, 48.5922, 51.6657, 54.7389, 57.8116, 60.884, 63.9559, 67.0272}.

Example 5.3. *Now, we will give an example connecting the zeros of the solutions to the differential/difference equation and the eignvalues of the corresponding Jacobi matrix.*

Let us consider the equation

$$-(py')' + qy = f$$

on $J = (a, b)$, where

$$J = (a, b) = (0, 1);\; k > 0;\; p(t) = \sin(t), \text{where } 0 < t < 1;\; q = 0;\; f = 0.$$

In this case, we have

$$-(py')' = 0.$$

Let $p_n = \sin(n/6)$, where $n \in [1, 5]$. Then,

$$\begin{aligned}
b_1 &= -2p_1 \approx -2 \cdot 0.1659 = -0.3318\\
b_2 &= -2p_2 \approx -2 \cdot 0.7401 = -1.4802\\
b_3 &= -2p_3 \approx -2 \cdot 0.4794 = -0.9588\\
b_4 &= -2p_4 \approx -2 \cdot 0.6184 = -1.2368\\
b_5 &= -2p_5 \approx -2 \cdot 0.7401 = -1.4802\\
a_1 &= -((1/2)p_{3/2} + p_1 - (1/2)p_{1/2}) \approx -1.1005\\
a_2 &= -((1/2)p_{5/2} + p_2 - (1/2)p_{3/2}) \approx -0.7098\\
a_3 &= -((1/2)p_{7/2} + p_3 - (1/2)p_{5/2}) \approx 0.3335\\
a_4 &= -((1/2)p_{9/2} + p_4 - (1/2)p_{7/2}) \approx 1.0702
\end{aligned}$$

and

$$J_5 = \begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & a_3 & 0 \\ 0 & 0 & a_3 & b_4 & a_4 \\ 0 & 0 & 0 & a_4 & b_5 \end{bmatrix}.$$

In this case, the eigenvalues of J_n are

$$-2.1799, -1.3264, -0.7394, \; -0.3344, -0.08197.$$

Now, we can calculate the corresponding $P_5(x)$. Using Mathematica and the recurrence relation derived above, we have

$$P_5(x) \approx (2.1799 + x)(1.3264 + x)(0.7394 + x)(0.3344 + x)(0.0820 + x).$$

Solving for $P_5(x) = 0$, we have the following solutions:

$$-2.1799, -1.3264, -0.7394, -0.3344, -0.0820.$$

Note that these solutions are very close to the eigenvalues for J, which verifies our construction above. They are slightly different due to the rounding errors in the calculations, especially the calculation of $P_5(x)$ using the recurrence relations, during which small rounding errors are multiplied and thus become more significant.

Example 5.4. *Below is a more complicated example connecting the zeros of the solutions to the differential/difference equation and the eignvalues of the corresponding Jacobi matrix. This example was also discussed in the previous section.*

Let us consider the equation

$$-(py')' + qy = f$$

on $J = (a, b)$, where

$$J = (a, b) = (0, 1);\ k > 0;\ p(t) = t^2/\sin(1/t), \text{where } 0 < t < 1, ;\ q = 0;\ f = 0.$$

Let $p_n = (n/6)^2/\sin(6/n)$, where $n \in [1, 5]$. Then,

$$\begin{aligned}
b_1 &= -2p_1 \approx 0.1988 \\
b_2 &= -2p_2 \approx -1.5747 \\
b_3 &= -2p_3 \approx -0.5499 \\
b_4 &= -2p_4 \approx -0.8911 \\
b_5 &= -2p_5 \approx -1.4902 \\
a_1 &= -((1/2)p_{3/2} + p_1 - (1/2)p_{1/2}) \approx 0.1342 \\
a_2 &= -((1/2)p_{5/2} + p_2 - (1/2)p_{3/2}) \approx -0.9572 \\
a_3 &= -((1/2)p_{7/2} + p_3 - (1/2)p_{5/2}) \approx -0.3183 \\
a_4 &= -((1/2)p_{9/2} + p_4 - (1/2)p_{7/2}) \approx -0.5630
\end{aligned}$$

and

$$J_5 = \begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & b_3 & a_3 & 0 \\ 0 & 0 & a_3 & b_4 & a_4 \\ 0 & 0 & 0 & a_4 & b_5 \end{bmatrix}.$$

In this case, the eigenvalues of J_n are

$$-2.1866, -1.8184, -0.6245, 0.0849, 0.2375.$$

Now, we can calculate the corresponding $P_5(x)$. Using Mathematica and the recurrence relation derived above, we have

$$P_5(x) \approx (-0.2375 + x)(-0.0849 + x)(0.6245 + x)(1.8184 + x)(2.1865 + x).$$

Solving for $P_5(x) = 0$, we have the following solutions:

$$-2.1866, -1.8184, -0.6245, 0.0849, 0.2375.$$

Note that these solutions are exactly the same as the eigenvalues for J, which verifies our construction above.

6 References

[1] A. Bielecki, *Une remarque sur la methode de Banach-Cacciopoli-Tikhonov dans la theorie des equations differentielles ordinaires*, Bull. Acad. Polon. Sci. Cl. III 4(1956), 261-264.

[2] S. Elaydi, *An Introduction to Difference Equations*, Second Edition, Springer-Verlag, New York, 1999.

[3] H.D. Niessen, A. Zettl, *Singular Sturm-Liouville problems: the Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc. 64 (1992) 545-578.

[4] Hermann Schulz-Baldes, *Sturm intersection theory for periodic Jacobi matrices and linear Hamiltonian systems*, unpublished.

[5] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.

[6] B. Simon, *Sturm oscillation and comparison theorems*, Sturm-Liouville Theory: Past and Present (W. Amrein, A. Hinz, and D. Pearson, eds.), pp. 29-43, Birkhaüuser, Basel, 2005.

[7] G. Szegö. *Orthogonal Polynomials*, 4th ed. Providence, RI: Amer. Math. Soc., pp. 44-47 and 54-55, 1975.

[8] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs 72, Amer. Math. Soc., Providence, RI, 2000.

[9] G. Teschl, *Topics in Real and Functional Analysis*, unpublished, available online at `http://www.mat.univie.ac.at/~gerald/`.

[10] W. Walter, *Ordinary differential equations, Graduate Texts in Mathematics*, Springer, New York, 1998.

[11] A. Zettl, *Sturm-Liouville Theory*, Amer. Math. Soc., Mathematical Surveys and Monographs 121, 2005.

Appendix

Calculations for the Prüfer Transformation Example

```
In[1]:= Clear[y]

In[2]:= p := 1 / (t ^ (-2) * Sin[1 / t])

In[3]:= q := 0

In[36]:= λ := 0

In[5]:= w := 0

In[6]:= DSolve[D[-(p * y'[t]), t] + q * y[t] == w * y[t], y[t], t]
```

Out[6]= $\left\{\left\{y[t] \to C[2] + C[1]\, \mathrm{Cos}\left[\frac{1}{t}\right]\right\}\right\}$

```
In[7]:= r := 1 / p

In[37]:= g := λ * w - q
```

In[12]:= $v := \mathrm{Cos}\left[\frac{1}{t}\right]$

```
In[14]:= v1 := FullSimplify[p * D[v, t]]

In[15]:= v1
```

Out[15]= 1

```
In[38]:= Solve[Tan[θ] == v, θ]
```

Out[38]= $\left\{\left\{\theta \to \mathrm{ArcTan}\left[\mathrm{Cos}\left[\frac{1}{t}\right]\right]\right\}\right\}$

In[40]:= $\theta := \mathrm{ArcTan}\left[\mathrm{Cos}\left[\frac{1}{t}\right]\right]$

```
In[42]:= D[θ, t]
```

Out[42]= $\dfrac{\mathrm{Sin}\left[\frac{1}{t}\right]}{t^2\left(1 + \mathrm{Cos}\left[\frac{1}{t}\right]^2\right)}$

```
r * (Cos[θ]) ^ 2
```

Out[19]= $\dfrac{\mathrm{Sin}\left[\frac{1}{t}\right]}{t^2\left(1 + \mathrm{Cos}\left[\frac{1}{t}\right]^2\right)}$

Calculations for the Difference/Differential Equations

```
In[522]:= f[n_] := Sin[n / 6];
b1 = N[(-2) * f[1]];
b2 = N[(-2) * f[2]];
b3 = N[(-2) * f[3]];
b4 = N[(-2) * f[4]];
b5 = N[(-2) * f[5]];
a1 = N[-(1 / 2 * f[3 / 2] + f[1] - 1 / 2 * f[1 / 2])];
a2 = N[-(1 / 2 * f[5 / 2] + f[2] - 1 / 2 * f[3 / 2])];
a3 = N[-(1 / 2 * f[7 / 2] + f[3] - 1 / 2 * f[5 / 2])];
a4 = N[-(1 / 2 * f[9 / 2] + f[4] - 1 / 2 * f[7 / 2])];
a5 = N[-(1 / 2 * f[11 / 2] + f[5] - 1 / 2 * f[9 / 2])];

In[533]:= u0 = 0;
u1 = 1;
u2 = a1^(-1) * (x - b1) * u1;
u3 = a2^(-1) * (x - b2) * u2 - a2^(-1) * a1 * u1;
u4 = a3^(-1) * (x - b3) * u3 - a3^(-1) * a2 * u2;
u5 = a4^(-1) * (x - b4) * u4 - a4^(-1) * a3 * u3;
u6 = a5^(-1) * (x - b5) * u5 - a5^(-1) * a4 * u4;

In[550]:= p0 = u1;
P0 = p0;
p1 = u2;
P1 = a1 * p1;
P2 = x * P1 - b2 * P1 - a1^2 * P0;
P3 = x * P2 - b3 * P2 - a2^2 * P1;
P4 = x * P3 - b4 * P3 - a3^2 * P2;
P5 = x * P4 - b5 * P4 - a4^2 * P3;

In[558]:= Simplify[P5]

0.9999999999999999` (0.08196965357438073` + x)
 (0.3344195990109977` + x) (0.7394167814645523` + x)
 (1.3264061277550918` + x) (2.179913886914066` + x)

In[559]:= Solve[P5 == 0, x]

{{x -> -2.1799138869140684`},
 {x -> -1.32640612775509`}, {x -> -0.7394167814645526`},
 {x -> -0.3344195990109978`}, {x -> -0.08196965357438073`}}

In[549]:= Eigenvalues[
 {{b1, a1, 0, 0, 0}, {a1, b2, a2, 0, 0}, {0, a2, b3, a3, 0}, {0, 0, a3, b4, a4}, {0, 0, 0, a4, b5}}]

{-2.1799138869140666`, -1.32640612775509`,
 -0.739416781464554`, -0.3344195990109972`, -0.08196965357438074`}
```

Calculations for the Difference/Differential Equations

```
In[380]:= f[n_] := (n / 6) ^ 2 / Sin[6 / n];
b1 = N[(-2) * f[1]];
b2 = N[(-2) * f[2]];
b3 = N[(-2) * f[3]];
b4 = N[(-2) * f[4]];
b5 = N[(-2) * f[5]];
a1 = N[-(1 / 2 * f[3 / 2] + f[1] - 1 / 2 * f[1 / 2])];
a2 = N[-(1 / 2 * f[5 / 2] + f[2] - 1 / 2 * f[3 / 2])];
a3 = N[-(1 / 2 * f[7 / 2] + f[3] - 1 / 2 * f[5 / 2])];
a4 = N[-(1 / 2 * f[9 / 2] + f[4] - 1 / 2 * f[7 / 2])];
a5 = N[-(1 / 2 * f[11 / 2] + f[5] - 1 / 2 * f[9 / 2])];

In[391]:= u0 = 0;
u1 = 1;
u2 = a1 ^ (-1) * (x - b1) * u1;
u3 = a2 ^ (-1) * (x - b2) * u2 - a2 ^ (-1) * a1 * u1;
u4 = a3 ^ (-1) * (x - b3) * u3 - a3 ^ (-1) * a2 * u2;
u5 = a4 ^ (-1) * (x - b4) * u4 - a4 ^ (-1) * a3 * u3;
u6 = a5 ^ (-1) * (x - b5) * u5 - a5 ^ (-1) * a4 * u4;

In[398]:= p0 = u1;
P0 = p0;
P1 = a1 * p1;
P2 = x * P1 - b2 * P1 - a1 ^ 2 * P0;
P3 = x * P2 - b3 * P2 - a2 ^ 2 * P1;
P4 = x * P3 - b4 * P3 - a3 ^ 2 * P2;
P5 = x * P4 - b5 * P4 - a4 ^ 2 * P3;

In[405]:= Simplify[P5]

1.` (-0.2375459596746414` + x) (-0.08490891578562153` + x)
 (0.6245388095387586` + x) (1.818400552959919` + x) (2.186549506241636` + x)

In[415]:= Solve[P5 == 0, x]

{{x → -2.1865495062416342`},
 {x → -1.8184005529599214`}, {x → -0.6245388095387583`},
 {x → 0.08490891578562156`}, {x → 0.2375459596746413`}}

In[419]:= Eigenvalues[
 {{b1, a1, 0, 0, 0}, {a1, b2, a2, 0, 0}, {0, a2, b3, a3, 0}, {0, 0, a3, b4, a4}, {0, 0, 0, a4, b5}}]

{-2.1865495062416325`, -1.818400552959922`,
 -0.624538809538759`, 0.23754595967464148`, 0.08490891578562179`}
```