Spectral Properties of Random Unitary Band Matrices

by

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A Thesis
Submitted in partial fulfillment
of the requirements for the Degree of Bachelor of Arts with Honors
in Mathematics

WILLIAMS COLLEGE

Williamstown, Massachusetts

May 09, 2007

It is a pleasure to thank Professor Stoiciu for his patience and selflessness in being my thesis advisor. Second, I would like to thank Professor Silva for being my second reader and for his helpful suggestions. Finally, I would like to thank my parents, without whom I could never have completed this thesis.

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Chapter 1

Preliminaries

1.1 Operator Spectra

One of the most important results in analysis is the Spectral Theorem, which shows the connection between linear operators on a Hilbert Space $\mathcal{H}$ and measures. This fundamental result illustrates a strong connection between algebraic objects (i.e. linear operators) and analytic objects (i.e. measures). This connection is a powerful idea in mathematics as it allows us to use tools from one area of math to make conclusions in another area. Our most powerful results will be rooted in this connection. We begin with a statement of the Spectral Theorem for unitary operators (taken from [7]).

Spectral Theorem. Let U be a unitary operator on a separable Hilbert Space $\mathcal{H}$. Let φ be a cyclic vector for U in $\mathcal{H}$. There exists a unique measure μ on the unit circle (or equivalently a measure on the interval $[0, 2\pi)$) such that for all $n \in \mathbb{Z}$ we have

$$\langle \varphi, U^n \varphi \rangle = \int_{\partial \mathbb{D}} z^n d\mu(z) = \int_0^{2\pi} e^{inz} d\mu(z).$$

Since the polynomials are dense in the space of $\mathcal{L}^2$ functions, this allows us to conclude that the same statement is true for any $\mathcal{L}^2$ function of U. That is, for any $f \in \mathcal{L}^2$ we have

$$\langle \varphi, f(U)\varphi \rangle = \int_{\partial \mathbb{D}} f(z) d\mu(z).$$

In particular, we have

$$\langle \varphi, (U - wI)^{-1} \varphi \rangle = \int_{\partial \mathbb{D}} \frac{1}{z - w} d\mu(z)$$

for $w \in \mathbb{C}$ provided the inverse exists (and we will see that sometimes it does not). We call $(U - wI)^{-1}$ the *resolvent* of U.

When we say that μ is a measure on the unit circle, this is a generalization that applies to all unitary operators. Given a particular operator unitary U_0, we can say that its spectral measure will be supported on the spectrum of U_0, where the spectrum of an operator is defined as follows.

Definition 1. *The spectrum of an operator U is the set of all complex numbers $z \in \mathbb{C}$ such that $(U - zI)$ is not invertible. We denote this set by $\sigma(U)$.*

Since the spectrum of a unitary operator is always contained in the unit circle, the spectral measure of a unitary operator is always a measure on the unit circle. The Spectral Theorem essentially tells us that we can obtain information about measures on the circle by studying the spectra of unitary operators. The following theorem of Verblunsky (several proofs of which can be found in [9]) shows us that this approach is even more meaningful that we might have originally anticipated.

Verblunsky's Theorem. There exists a one to one correspondence between sequences of numbers in $\mathbb{D}$ and nontrivial (i.e. not supported on a finite set) probability measures on the unit circle.

This correspondence can be realized in a natural way by using two objects that will be of central importance to us. The first is *Orthogonal Polynomials on the Unit Circle* (OPUC) and the second is CMV (Cantero, Moral, and Velázquez) matrices. The way we can see this correspondence is shown here.

$$\mu \to OPUC \to \{\alpha_n\}_{n\geq 0} \to CMV \to \mu$$

Suppose we start with a measure μ on the unit circle and the set of polynomials $\{1, x, x^2, x^3, \ldots\}$. We can perform the Gram-Schmidt Orthogonalization process to this set (with respect to the $\mathcal{L}^2$ norm) to get a new sequence of orthogonal polynomials $\{1, \Phi_1, \Phi_2, \Phi_3, \ldots\}$. These polynomials satisfy the recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z)$$

where if

$$\Phi_n(z) = \sum_{j=0}^{n} b_j z^j$$

then

$$\Phi_n^*(z) = \sum_{j=0}^{n} \bar{b}_{n-j} z^j.$$

Each of the complex numbers α_n, which we call Verblunsky coefficients, is contained in $\mathbb{D}$.

Conversely, if we start with a sequence $\{\alpha_n\}_{n\geq 0} \in \mathbb{D}^{\mathbb{N}}$, then we can fill in the CMV matrix shown here.

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\rho_k = \sqrt{1-|\alpha_k|^2}.$$

The CMV matrix is a five diagonal unitary matrix meaning all of the nonzero matrix elements are located on the main diagonal, the two diagonals below it, and the two diagonals above it and it is unitary. Since it is unitary, the spectral measure associated to this matrix will be supported on $\partial\mathbb{D}$ so this gives us a nontrivial measure on the circle. Notice that if one of the α_j has absolute value 1, then the matrix decouples. The upper left corner is then an $n \times n$ unitary matrix, which we denote by $\mathcal{C}^{(n)}$ and the eigenvalues of $\mathcal{C}^{(n)}$ are exactly the zeros of the paraorthogonal polynomial Φ_n (paraorthogonal means $|\alpha_n| = 1$).

Verblunsky's Theorem is a beautiful result. It puts four seemingly unrelated objects - measures, orthogonal polynomials, sequences in $\mathbb{D}$, and CMV matrices - in one to one correspondence with one another. Therefore, by studying one of these objects, we can gain information about the other three. Furthermore, by means of this correspondence we can use tools meant for studying one object (measures for example) to study another (such as orthogonal polynomials). The primary focus of this work will be to study the spectral properties of CMV matrices and "Joye" matrices, another kind of unitary band matrix.

1.2 CMV Matrices

Much expository work on CMV matrices has been done in [9]. We will summarize some of the important results and give a few examples. To begin, let us derive the form of the CMV matrix. For a given measure μ, let us start with the set $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$, which forms a basis for $\mathcal{L}^2(\partial\mathbb{D}, d\mu)$. We can perform the Graham-Schmidt Orthogonalization process to get a new basis of orthogonal functions, which we denote by $\{\chi_0, \chi_1, \chi_2, \dots\}$. The CMV matrix is defined by

$$\mathcal{C}_{ij}(d\mu) = \langle \chi_i, z\chi_j \rangle$$

that is, the CMV matrix is the "multiplication by variable" operator on the space $\mathcal{L}^2(\partial\mathbb{D}, d\mu)$ with respect to the basis $\{\chi_j\}_{j\geq 0}$.

One of the most striking properties of a CMV matrix is that it can be factored. Let us define the matrix Θ_j by

$$\Theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\bar{\alpha}_j \end{pmatrix}.$$

Now we can define the matrices $\mathcal{M}$ and $\mathcal{L}$ by

$$\mathcal{M} = \begin{pmatrix} 1 & & & \\ & \Theta_1 & & \\ & & \Theta_3 & \\ & & & \ddots \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} \Theta_0 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix}.$$

One can easily verfiy that $\mathcal{C} = \mathcal{L}\mathcal{M}$. In our original construction of the CMV matrix, we could have started with the set $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$ and proceeded in the same way. This procedure would have resulted in a different basis of orthogonal functions, which we denote by $\{x_j\}_{j\geq 0}$. It is interesting to note that

$$\mathcal{M}_{ij}(d\mu) = \langle x_i, \chi_j \rangle, \quad \mathcal{L}_{ij}(d\mu) = \langle \chi_i, z x_j \rangle.$$

The following example is taken from [9].

Example 1. *If we let* $d\mu = \frac{d\theta}{2\pi}$*, that is, if we consider normalized Lebesgue measure on the circle, then all of the Verblunsky coefficients are* 0 *(i.e.* $\alpha_j = 0$ *for all* $j \geq 0$*). Therefore,*

$$\Theta_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $j \geq 0$ *and it follows that*

$$\mathcal{M} = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & \ddots \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

and therefore

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 1 & & & \\ 1 & 0 & 0 & & & \\ & 0 & 0 & 0 & 1 & \\ & 1 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & & \ddots \end{pmatrix}.$$

This particular example is called the "free case" indicating that the spectral measure corresponding to $\mathcal{C}$ has no pure points.

1.3 Applications

The spectral properties of unitary and self-adjoint operators are very important in Mathematical Physics. Here we briefly discuss one particular application of Joye matrices. In [1], Blatter and Browne investigate a phenomenon called *Zener Tunneling*. *Zener Tunneling* is a model of the behavior of an electron in a metal ring that is threaded with a time dependent magnetic flux, which induces an electric current that increases linearly with time. The model used in [1] shows that at certain times during this "ramping up" of the magnetic field, an electron in a bound state can tunnel out of its state into a neighboring state with probability $t^2 < 1$. We can interpret t as a transmission coefficient, and define a reflection coefficient r so that $r^2 + t^2 = 1$. Therefore, the matrix that represents the time evolution of an electron in a bound state is

$$S^+ = \begin{pmatrix} -r & rt & -t^2 & 0 & 0 & \\ t & r^2 & -rt & 0 & 0 & \\ 0 & rt & r^2 & rt & -t^2 & \\ 0 & -t^2 & rt & r^2 & -rt & \\ 0 & 0 & 0 & rt & r^2 & \\ 0 & 0 & 0 & -t^2 & -rt & \ddots \end{pmatrix}$$

which is clearly a five diagonal unitary band matrix. In [1], the authors give each row a random phase (i.e. row m is multiplied by $e^{i\theta_m}$ for some random θ_m) and perform numerical simulations to show that the eigenfunctions of this operator - which represent eigenstates to a physicist - are exponentially localized in energy space. The final result of this thesis is devoted to proving that this is in fact the case provided the distribution of the phases satisfies a set of weak conditions.

Another application of these methods to physical systems can be found in [2]. In this paper, Combescure uses methods similar to those in [3] to study time dependent Hamiltonians in quantum systems. We will return to his work in more detail in Section 4.6.

Chapter 2

Introduction

We now define some notation that we will use throughout the first three Chapters of this thesis. Define the space Ω as in [13] by

$$\Omega = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}) \in \mathbb{D}(0,R) \times \mathbb{D}(0,R) \times \ldots \times \mathbb{D}(0,R) \times \partial\mathbb{D}\}$$

with the probability measure $\mathbb{P}$ obtained by taking the product of the probability measure $d\mu$ on each $\mathbb{D}(0,R)$ and uniform Lebesgue measure on $\partial\mathbb{D}$. We will denote by $\mathcal{C}_\alpha^{(n)}$ the truncated CMV matrix $\mathcal{C}^{(n)}$ corresponding to some given $\alpha \in \Omega$. We will also use

$$F_{kl}(z, \mathcal{C}_\alpha^{(n)}) = [(\mathcal{C}_\alpha^{(n)} + z)(\mathcal{C}_\alpha^{(n)} - z)^{-1}]_{kl}$$

and

$$G_{kl}(z, \mathcal{C}_\alpha^{(n)}) = [(\mathcal{C}_\alpha^{(n)} - z)^{-1}]_{kl}.$$

Our first new result is in the spirit of the result presented in [13] and our proof follows the road map presented there. We would like to prove results about the spectral properties of random CMV matrices when the Verblunsky coefficients are randomized in a particular fashion. In [13], Stoiciu showed that that if the Verblunsky coefficients are i.i.d. random variables uniformly distributed in the disk of radius $R < 1$, then the asymptotic distribution of the eigenvalues of the corresponding CMV matrix is almost surely Poisson (i.e. they exhibit no correlation). The difference between this result and Stoiciu's result from [13] is that in [13], the probability distribution $\mathbb{P}$ of the Verblunsky coefficients was the product of uniform Lebesgue measure on $\mathbb{D}(0,R)$ whereas here, we will use the measure $\gamma\frac{\chi_r d\lambda(z)}{|z|^\sigma}$ with $0 < \sigma \leq 1$, χ_r indicating that the support of the measure is the disk of radius $r < 1$, and $\gamma \equiv \frac{2-\sigma}{2\pi r^{2-\sigma}}$ is the normalization constant to make our measure a probability measure (we will keep this definition of γ throughout). Shown here in Figure 2.1 is the distribution for $\sigma = 0.5$ and $r = 0.9$.

We will obtain the desired result by showing (steps taken from [13])

1. (Fractional Moment Estimates a.k.a. Aizenman-Molchanov bounds) For the probability space Ω defined above, and for any $s \in (0,1)$, there exist constants $C_1, D_1 > 0$ that depend only on s such that for any $n > 0$, and k, l satisfying $0 \leq k, l \leq n-1$ and any $e^{i\theta} \in \partial\mathbb{D}$ we have

$$\mathbb{E}(|F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})|^s) \leq C_1 e^{-D_1|k-l|}.$$

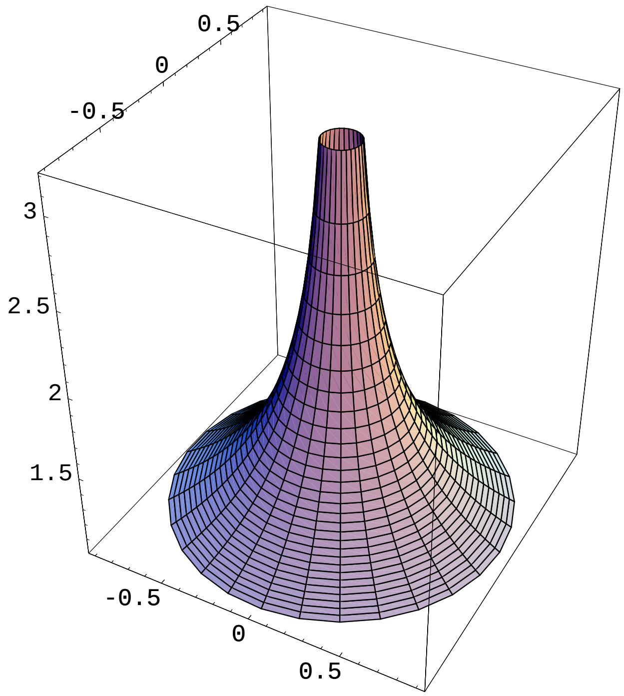

Figure 2.1: A new Distribution.

2. (Localization of Eigenfunctions) There exists a constant $D_2 > 0$ and for almost every $\alpha \in \Omega$ there exists a constant $C_\alpha > 0$ such that for any unitary eigenfunction $\varphi_\alpha^{(n)}$ of $\mathcal{C}_\alpha^{(n)}$, there exists a point $m(\varphi_\alpha^{(n)})$ with $1 \leq m(\varphi_\alpha^{(n)}) \leq n$ such that for any m satisfying $|m - m(\varphi_\alpha^{(n)})| \geq D_2 \ln(n+1)$, we have

$$|\varphi_\alpha^{(n)}(m)| \leq C_\alpha e^{-(4/D_2)|m - m(\varphi_\alpha^{(n)})|}$$

where we call the point $m(\varphi_\alpha^{(n)})$ the center of localization.

3. (Decoupling the Point Process) The point process $\zeta^{(n)} = \sum_{k=1}^{n} \delta_{z_k^{(n)}}$ (where the collection $\{z_k^{(n)}\}$ is the collection of eigenvalues of $\mathcal{C}^{(n)}$) can be asymptotically approximated by the direct sum of point processes $\sum_{p=1}^{[ln\, n]} \zeta^{(n,p)}$. That is, the distribution of the eigenvalues of $\mathcal{C}^{(n)}$ can be asymptotically approximated by the distribution of the eigenvalues of the direct sum of smaller matrices.

The motivation for considering the distribution pictured above is as follows. In [6], Killip and Stoiciu show that if the n^{th} Verblunsky coefficient is chosen from the uniform distribution on the disk of radius $n^{-\nu}$ with $\nu > 0.5$ then the asymptotic distribution of the eigenvalues of the corresponding CMV matrix is not Poisson, but is called “clock.” That is, the eigenvalues are asymptotically evenly spaced points on the unit circle. Since the Verblunsky coefficients decay to the origin in this result from [6], this motivated the idea of considering identically distributed Verblunsky coefficients chosen from a distribution that is highly concentrated near the origin. However, since we consider Verblunsky coefficients that are identically distributed, given any small $\epsilon > 0$, infinitely many of the Verblunsky coefficients will have norm greater than $R - \epsilon$ with probability one. Therefore, even though the distribution we consider is highly concentrated near the origin, we do not have decaying coefficients as in [6]. This leads us to the following conjecture.

Conjecture 1. *If the Verblunksy coefficients are chosen randomly from the distribution*

$$d\mu = \frac{2-\sigma}{2\pi r^{2-\sigma}} \frac{d\lambda(z)}{|z|^{\sigma}}$$

on the disk of radius $r < 1$ *with* $0 < \sigma < 2$*. then the asymptotic distribution of the eigenvalues of the corresponding CMV matrix will exhibit poisson statistics.*

We consider only the cases $0 < \sigma < 2$ because for $\sigma \geq 2$, the distribution is no longer normalizable. We prove that this conjecture is true for $0 < \sigma \leq 1$ in Chapter 3.

The proof of the above conjecture for $0 < \sigma \leq 1$ reveals to us the connection between the spectral properties of the CMV matrix and the way in which we randomize the Verblunsky coefficients. In general, the proofs will rely on one or many of the following properties of the distribution of the Verblunsky coefficients:

1. A functional form of the distribution,

2. The rotation invariance of the distribution,

3. The support of the distribution,

4. The lack of atoms in the distribution.

The second part of this thesis (Chapters 4, 5, and 6) is an attempt to weaken the dependence of our results on any of the last three conditions listed above; in particular, we will try to eliminate Property 2, rotation invariance. To do this, we will use a different type of random unitary band matrices that we will call *"Joye matrices"*. These matrices will be very useful tools for us since the matrix elements are randomized in a different way than the CMV matrix elements. A result by Joye et al. (see [3] and [4]) shows that under certain conditions, the expected value of the fractional moments of the matrix elements of the resolvent decays exponentially along the rows (i.e. Aizenman-Molchanov bounds) even if the distribution of the matrix elements is not rotation invariant. Chapter 4 gives a detailed explanation of the methods used by Joye et al. to obtain these estimates. Chapter 5 contains results about the spectra of Joye matrices when the parameters are randomized in certain ways.

The main attraction of Joye's results is that the angular distribution of the random parameters does not have to be constant on $\mathbb{T}$ (where $\mathbb{T}$ is the compactification of $\mathbb{R}$ mod 2π). The major shortcoming of his results is that they do not go as far as to completely characterize the distribution of the eigenvalues of the random matrices

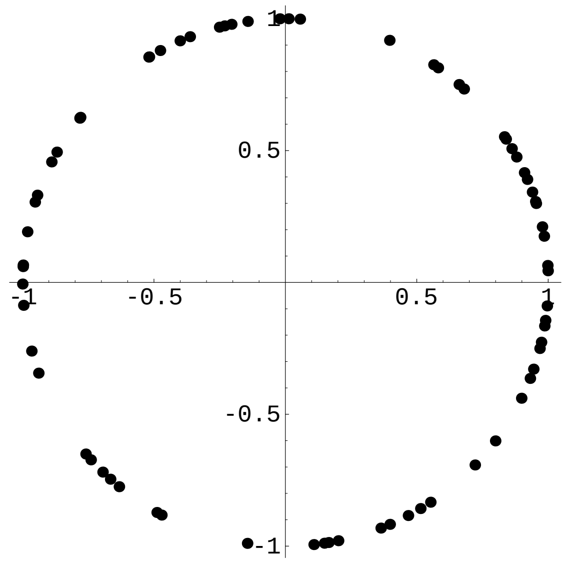

Figure 2.2: Eigenvalue Distribution showing Poisson Statistics.

(e.g. Poisson statistics). Numerical plots suggest that the asymptotic distribution of the eigenvalues of random Joye matrices is (almost surely) Poisson when t is small. Shown here in Figure 2.2 is the Mathematica plot of 71 eigenvalues of a random 71×71 Joye matrix when the phases are chosen from the uniform distribution on $[0, 5\pi)$.

The results in Chapter 6 bring us one step closer to classifying the distribution of eigenvalues. We show that Aizenman's Theorem holds true for random semi-infinite Joye matrices when the distribution of the phases is any one of an enormous class of distributions. That is, if $U^+(\omega)$ is a random semi-infinite Joye matrix, then if the phases are randomized appropriately, we have that

$$\mathbb{E}(\sup_n |[(U^+(\omega))^n]_{jk}|) \leq K_1 e^{-\gamma_1 |j-k|}$$

for appropriately chosen constants $K_1, \gamma_1 > 0$ independent of the random variable ω. Then, following the methods in [13], we can show that with probability 1, the eigenfunctions of $U^+(\omega)$ are exponentially localized. This is step 2 towards proving the existence of Poisson statistics (see above).

Chapter 3

Poisson Statistics for a New Distribution

3.1 Aizenman-Molchanov Bounds

It is our task to show that for any $s \in (0,1)$, there exist constants $C_1, D_1 > 0$ that depend only on s such that for any $n > 0$, and k, l satisfying $0 \leq k, l \leq n-1$ and any $e^{i\theta} \in \partial\mathbb{D}$ we have

$$\mathbb{E}(|F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})|^s) \leq C_1 e^{-D_1|k-l|}.$$

We proceed as in [13]. The first lemma we need is the following, taken from [13].

Lemma 3.1.1. [13] *For any $s \in (0,1)$ and any k, l satisfying $1 \leq k, l \leq n$ and any $z \in \mathbb{D} \cup \partial\mathbb{D}$ we have*

$$\mathbb{E}(|F_{kl}(z, \mathcal{C}_\alpha^{(n)})|^s) \leq C$$

where the constant C depends only on s.

Outline of Proof: After fixing $\rho \in (0,1)$, Kolmogorov's Theorem gives

$$\int_0^{2\pi} \left|(\varphi, (\mathcal{C}_\alpha^{(n)} + \rho e^{i\theta})(\mathcal{C}_\alpha^{(n)} - \rho e^{i\theta})^{-1})\right|^s \frac{d\theta}{2\pi} \leq C_1 = \frac{1}{\cos(\frac{\pi s}{2})}.$$

The polarization identity then gives us

$$\int_0^{2\pi} \left|F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)})\right|^s \frac{d\theta}{2\pi} \leq C = \frac{2^{2-s}}{\cos(\frac{\pi s}{2})}$$

Since the distribution if the α_k is rotationally invariant, we can consider a new set of Verblunsky coefficients $\alpha_{k,\theta} = e^{i(k+1)\theta}\alpha_k$ and use the reasoning in [13] to conclude that the function

$$\theta \to \mathbb{E}\left(\left|F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)})\right|^s\right)$$

is constant and the desired conclusion follows for $z \in \mathbb{D}$. Properties of Hardy spaces cited in [13] allow us to conclude that for Lebesgue a.e. $e^{i\theta} \in \partial\mathbb{D}$, the radial limit of $F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)})$ exists. The desired conlcusion for $z \in \partial\mathbb{D}$ follows from Fatou's Lemma.

Next we have the following proposition, which will be useful for us later.

Proposition 3.1.1. *Consider the probability measure given by* $d\mu_\alpha(\omega) = \gamma\frac{\chi_R d\lambda(\omega)}{|\omega|^\sigma}$ *(where* γ *is just a normalization constant) supported on a disk of radius* $R < 1$ *where* $0 < \sigma < 2$*. Then*

$$\int_{\mathbb{D}(0,1)} -\log(|\omega|)d\mu_\alpha(\omega) < \infty.$$

Proof. We have that

$$\begin{aligned}\int_{\mathbb{D}(0,1)} -\log(|\omega|)d\mu_\alpha &= \gamma\int_0^{2\pi}\int_0^R \frac{-\log(r)}{r^\sigma} r\,dr\,d\theta \\ &= 2\pi\gamma\int_0^R \frac{-\log(r)}{r^{\sigma-1}}dr.\end{aligned}$$

Clearly this integral diverges if $\sigma > 2$. Now, suppose $\sigma = 2 - \delta$. If $\delta = 0$ we have

$$\begin{aligned}\int_0^R \frac{-\log(r)}{r^{\sigma-1}}dr &= \int_0^R \frac{-\log(r)}{r}dr \\ &= -\frac{1}{2}\log^2(r)\Big|_0^R = \infty\end{aligned}$$

so the integral diverges.

If $\delta > 0$ then we make the substitution $y = \frac{1}{r}$ to get

$$\begin{aligned}\int_0^R \frac{-\log(r)}{r^{\sigma-1}}dr &= \int_\infty^{\frac{1}{R}} \frac{(-\log(y))y^{1-\delta}}{y^2}dy \\ &= \int_{\frac{1}{R}}^\infty \frac{\log(y)}{y^{1+\delta}}dy \\ &< \int_{\frac{C_0}{R}}^\infty \frac{1}{y^{1+\delta/2}}dy < \infty\end{aligned}$$

for some constat C_0 so the integral converges as desired. □

Next we will prove that

$$\int_{\mathbb{D}(0,R)} -\log(1-|\omega|)d\mu_\alpha(\omega) < \infty.$$

Let us define the region A as follows:

$$A = \{z : \frac{1}{2} < |z| \leq R\}.$$

That is, A is the annulus with inner radius $\frac{1}{2}$ and outer radius the same as the support of $d\mu_\alpha(\omega)$ (if $R < \frac{1}{2}$ then we define $A = \varnothing$).

Proposition 3.1.2. *Let* $f(z) : \mathbb{D}(0,R) \to \mathbb{R}^+$ *be a positive rotationally symmetric integrable function such that there exists a constant* C *with* $0 < f(z) < C$ *for a.e.* $z \in A$. *Also suppose that*

$$\int_{\mathbb{D}(0,R)} -\log(|z|)f(z)d\lambda(z) < \infty.$$

Then

$$\int_{\mathbb{D}(0,R)} -\log(1-|z|)f(z)d\lambda(z) < \infty.$$

This proposition says that as long as the distribution from which we are drawing the Verblunsky coefficients is bounded outside the disk of radius $\frac{1}{2}$ then it suffices to check that

$$\int_{\mathbb{D}(0,R)} -\log(|z|)f(z)d\lambda(z) < \infty$$

to show the finiteness of both integrals.

Proof. We clearly have that

$$\begin{aligned}\int_{\mathbb{D}(0,R)} -\log(1-|z|)f(z)d\lambda(z) &= \int_{D(0,\frac{1}{2})} -\log(1-|z|)f(z)d\lambda(z) \\ &+ \int_A -\log(1-|z|)f(z)d\lambda(z).\end{aligned}$$

Since $f(z)$ and $-\log(1-|z|)$ are both bounded and positive in the region A, their product is also bounded and positive, so the integral of their product over the region A is finite. Thus we have

$$\int_A -\log(1-|z|)f(z)d\lambda(z) < \infty.$$

To deal with the other part of the integral, note that

$$\begin{aligned} & -\log(1-|z|)f(z) \leq -\log(|z|)f(z) \\ \iff & \log(1-|z|) \geq \log(|z|) \\ \iff & 1-|z| \geq |z| \\ \iff & |z| \leq \frac{1}{2}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{D}(0,\frac{1}{2})} -\log(1-|z|)f(z)d\lambda(z) < \int_{\mathbb{D}(0,\frac{1}{2})} -\log(|z|)f(z)d\lambda(z) < \infty.$$

It follows immediately that

$$\int_{\mathbb{D}(0,R)} -\log(1-|z|)f(z)d\lambda(z) < \infty$$

as desired. □

From this, we get the following corollary

Corollary 3.1.2. *Consider the measure given by* $d\mu_\alpha(\omega) = \gamma \frac{\chi_R d\lambda(\omega)}{|\omega|^\sigma}$ *supported on a disk of radius* $R < 1$ *where* $0 < \sigma < 2$*. Then*

$$\int_{\mathbb{D}(0,1)} -\log(1 - |\omega|) d\mu_\alpha(\omega) < \infty.$$

Proof. It is easily seen that $\frac{1}{|z|^\sigma}$ with $0 < \sigma < 2$ satisfies the conditions of Lemma 3.1.2 and by Lemma 3.1.1 we know that

$$\int_{\mathbb{D}(0,1)} -\log(|\omega|) d\mu_\alpha(\omega) < \infty.$$

Therefore,

$$\int_{\mathbb{D}(0,1)} -\log(1 - |\omega|) d\mu_\alpha(\omega) < \infty$$

as desired. □

We can now proceed with the proof of the following lemma. Recall the definition of $G_{j,j+k}(z, \mathcal{C})$ from Chapter 2.

Lemma 3.1.3. [13] *Let* $\mathcal{C} = \mathcal{C}_\alpha$ *be the random CMV matrix associated to a family of Verblunsky coefficients* $\{\alpha_n\}_{n\geq 0}$ *with* α_n *i.i.d. random variables distributied on the disk* $\mathbb{D}(0, R)$ *according to the distribution* $\gamma \frac{\chi_r d\lambda(z)}{|z|^\sigma}$ *with* $r < 1$*. Let* $s \in (0, 1)$*,* $z \in \mathbb{D} \cup \partial\mathbb{D}$*, and* j *a positive integer. Then we have*

$$\lim_{k\to\infty} \mathbb{E}(|G_{j,j+k}(z, \mathcal{C})|^s) = 0.$$

Proof. The proof presented here follows that presented with Lemma 3.3 in [13].

The proof for $z \in \mathbb{D}$ is easy and is given in [13]. Now consider $z = e^{i\theta} \in \partial\mathbb{D}$. The transfer matrices corresponding to the CMV matrix are

$$T_n(z) = A(\alpha_n, z), \ldots, A(\alpha_0, z)$$

where

$$A(\alpha, z) = (1 - |\alpha|^2)^{-1/2} \begin{pmatrix} z & -\overline{\alpha} \\ -\alpha z & 1 \end{pmatrix}$$

and the Lyapunov exponent is

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log ||T_n(z, \{\alpha_n\}||$$

(provided this limit exists). Observe that the common distribution $d\mu_\alpha$ of the Verblunsky coefficients is rotationally invariant and

$$\int_{\mathbb{D}(0,1)} -\log(1 - |\omega|) d\mu_\alpha(\omega) < \infty$$

and

$$\int_{\mathbb{D}(0,1)} -\log(|\omega|)d\mu_\alpha(\omega) < \infty$$

by 3.1.1 and 3.1.2.

By rotation invariance, the density of eigenvalues measure is just $\frac{d\theta}{2\pi}$ and therefore the logarithmic potential of this measure is identically zero. The Lyapunov exponent exists for every $z = e^{i\theta} \in \partial\mathbb{D}$ and the Thouless formula gives

$$\gamma(z) = -\frac{1}{2}\int_{\mathbb{D}(0,1)} \log(1-|\omega|^2)d\mu_\alpha(\omega).$$

It is easily seen that

$$-\frac{1}{2}\int_{\mathbb{D}(0,1)} \log(1-|\omega|^2)d\mu_\alpha(\omega) > -\frac{1}{2}\int_{\mathbb{D}(0,1)} \log(1-|\omega|^2)\frac{d\theta}{\pi R^2}(\omega) > 0$$

where $\frac{d\theta}{\pi R^2}(\omega)$ represents the uniform distribution on $\mathbb{D}(0,R)$ and for the last inequality we used the result of [13]. It follows then that the Lyapunov exponent $\gamma(e^{i\theta})$ is positive, and using the Ruelle-Osceledec Theorem, we conclude that there exists a constant $\lambda \neq 1$ for which

$$\lim_{n\to\infty} T_n(e^{i\theta})\begin{pmatrix}1\\ \lambda\end{pmatrix} = 0.$$

From here, we use the same reasoning as in [13] (i.e. the theory of subordinate solutions) to conclude that for any j and almost every $e^{i\theta} \in \partial\mathbb{D}$,

$$\lim_{k\to\infty} G_{j,j+k}(e^{i\theta},\mathcal{C}) = 0.$$

the desired conclusion follows as in [13]. □

We will also need the following lemma from [13].

Lemma 3.1.4. [13] *For any fixed j, and $s \in (0,1)$, and any $z \in \mathbb{D}$,*

$$\lim_{k\to\infty,\, k\le n} \mathbb{E}\left(\left|G_{j,j+k}\left(z,\mathcal{C}_\alpha^{(n)}\right)\right|^s\right) = 0.$$

The key to the proof of this lemma is to form the matrix $\mathcal{C}^{(n)}$ by decoupling the matrix $\mathcal{C}$ and apply the resolvent identity to $(\mathcal{C}-\mathcal{C}^{(n)})$, a matrix with at most eight nonzero terms. Applying the same decoupling trick, we get the following lemma from [13].

Lemma 3.1.5. [13] *For any $\epsilon > 0$, there exists a $k_\epsilon \ge 0$ such that for any $s \in (0,1)$ and $k > k_\epsilon$ and $n > 0$ and $0 \le j \le (n-1)$ and for any $z \in \mathbb{D}\cup\partial\mathbb{D}$, we have*

$$\mathbb{E}\left(\left|G_{j,j+k}\left(z,\mathcal{C}_\alpha^{(n)}\right)\right|^s\right) < \epsilon.$$

Following the proof in [13], we have the following lemma. It is interesting to note that this lemma is where we most heavily rely on the fact that the support of our distribution is the disk of radius $R < 1$.

Lemma 3.1.6. [13] *Let $\mathcal{C}_\alpha^{(n)}$ be exactly as defined before. Then, for any $e^{i\theta} \in \partial\mathbb{D}$ and for any $\alpha \in \Omega$ (defined above) where $G(e^{i\theta}, \mathcal{C}_\alpha^{(n)}) = (\mathcal{C}_\alpha^{(n)} - e^{i\theta})^{-1}$ exists, we have*

$$\frac{|G_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})|}{|G_{ij}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})|} \leq \left(\frac{2}{\sqrt{1-R^2}}\right)^{|k-i|+|l-j|}.$$

The key to the proof of this lemma is the identity provided in [9], which shows that

$$\left[(\mathcal{C}-z)^{-1}\right]_{kl} = \begin{cases} (2z)^{-1}(\chi_l(z)p_k(z), & k > l, \quad \text{or} \quad k = l = 2n-1, \\ (2z)^{-1}(\pi_l(z)x_k(z), & l > k, \quad \text{or} \quad k = l = 2n, \end{cases}$$

where $\chi_l(z)$ and $x_k(z)$ are orthogonal polynomials obtained from applying Gram-Schmidt to $\{1, z, z^{-1}, z^2, \ldots\}$ and $\{1, z^{-1}, z, z^{-2}, \ldots\}$ respectively and $p_l(z)$ and $\pi_k(z)$ are analogs of the Weyl solutions of Golinskii-Nevai ([13]).

Our next lemma is in the spirit of Lemma 3.8 from [13], but modified so that we can apply it to our measure $d\mu$.

Lemma 3.1.7. *For any constant $s \in (0,1)$ and any constant $\beta \in \mathbb{C}$ and any $\sigma \in [0,1]$ and any $y \in [-1,1]$, we have*

$$\int_{-1}^{1} \frac{1}{|x-\beta|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \leq \int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx.$$

Proof. Let $\beta = \beta_1 + i\beta_2$ with $\beta_1, \beta_2 \in \mathbb{R}$ and let us assume without loss of generality that $\beta_1 \geq 0$. Then

$$\begin{aligned} \int_{-1}^{1} \frac{1}{|x-\beta|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx &= \int_{-1}^{1} \frac{1}{|(x-\beta_1)^2 + \beta_2^2|^{s/2}} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \\ &\leq \int_{-1}^{1} \frac{1}{|x-\beta_1|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx. \end{aligned}$$

If $\beta_1 = 0$ then we are done. Otherwise, consider the expression

$$\begin{aligned} &\int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx - \int_{-1}^{1} \frac{1}{|x-\beta_1|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \\ &= \int_{-1}^{1} \frac{|x-\beta_1|^s - |x|^s}{|x-\beta_1|^s|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx. \end{aligned}$$

Notice that the expression $g(x) \equiv \frac{|x-\beta_1|^s - |x|^s}{|x-\beta_1|^s|x|^s}$ is positive on the interval $[-1, \beta_1/2)$ and negative on the interval $(\beta_1/2, 1]$. Suppose that $g(x)$ is negative at some $x_0 = \beta_1/2 + \delta \leq 1$. Then at $x_1 = \beta_1/2 - \delta$ the expression is positive, but equal in absolute value (that is $|g(x_0)| = |g(x_1)|$). However, $\frac{1}{|x_1^2+y^2|^{\sigma/2}} > \frac{1}{|x_0^2+y^2|^{\sigma/2}}$, that is, our measure $\frac{1}{|x^2+y^2|^{\sigma/2}}$ applies more weight to the positive part (recalling that y is fixed). Therefore, if we let $\epsilon = 1 - \beta_1/2$ then

$$\begin{aligned} &\int_{-1}^{1} \frac{|x-\beta_1|^s - |x|^s}{|x-\beta_1|^s|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \\ \geq &\int_{-1}^{1-2\epsilon} \frac{|x-\beta_1|^s - |x|^s}{|x-\beta_1|^s|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \geq 0 \end{aligned}$$

since we have assumed that $\epsilon < 1$. Therefore, we conclude that

$$\int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx - \int_{-1}^{1} \frac{1}{|x-\beta_1|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \geq 0$$
$$\Rightarrow \int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \geq \int_{-1}^{1} \frac{1}{|x-\beta_1|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx$$
$$\Rightarrow \int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx \geq \int_{-1}^{1} \frac{1}{|x-\beta|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx$$

for any complex number β as desired. □

We will apply this lemma by means of the following corollary. First though, we must define two functions. For any $D, E \in \mathbb{C}$, let us define $f_{D,E} : [-1,1] \to \mathbb{R}^+ \cup \{\infty\}$ by

$$f_{D,E}(y) = \int_{-1}^{1} \frac{1}{|x+yD+E|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx$$

and let us define $f : [-1,1] \to \mathbb{R}^+ \cup \{\infty\}$ by

$$f(y) = \int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{(x^2+y^2)^{\sigma/2}} dx.$$

With this notation, we have the following corollary.

Corollary 3.1.8. *For any $s \in (0,1)$ and any $\sigma \in [0,1]$ and any $y \in [-1,1]$ and any $D, E \in \mathbb{C}$ we have $f_{D,E}(y) \leq f(y)$.*

Proof. Since y, D, and E are fixed, we can apply Lemma 3.1.7 with $-\beta = yD + E$ to get the desired conclusion. □

From this, we get the following corollary, which we will apply directly to achieve a desired lemma.

Corollary 3.1.9. *For any $s \in (0,1)$ and any $\sigma \in [0,1]$ and any $D, E \in \mathbb{C}$ and with our definitions of $f(y)$ and $f_{D,E}(y)$ as before, we have*

$$\int_{-1}^{1} f_{D,E}(y) dy \leq \int_{-1}^{1} f(y) dy.$$

Proof. Corollary 3.1.8 shows that $f_{D,E}(y) \leq f(y)$ on the interval $[-1,1]$. The desired conclusion follows immediately. □

It will also be helpful to have the following, the proof of which is straightforward.

Proposition 3.1.3. *For any $s \in (0,1)$ we have that*

$$\int_{0}^{\pi/2} \csc(x)^s dx < C_s$$

where C_s is a constant that depends only on the value of s.

Now we can prove the desired lemma, which is a new version of Lemma 3.9 from [13].

Lemma 3.1.10. *For any $s \in (0,1)$, and k satisfying $1 \le k \le n$, and any choice of $\alpha_0 \ldots \alpha_{k-1}, \alpha_{k+1} \ldots, \alpha_{n-1}$,*

$$\mathbb{E}(|F_{kk}(z, \mathcal{C}_\alpha^{(n)})|^s | \{\alpha_i\}_{i \neq k}) \le C$$

for some constant C that may depend on s and σ.

Proof. From Lemma 3.9 of [13], we know that the diagonal elements we wish to bound are given by

$$(\delta_k, (\mathcal{C}+z)(\mathcal{C}-z)^{-1}\delta_k) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z} |\varphi(e^{i\theta})|^2 d\mu(e^{i\theta})$$

where μ is the measure associated with the collection of Verblunsky coefficients $\{\alpha_n\}_{n\ge 0}$ and $\{\varphi_n\}_{n \ge 0}$ are the resulting orthonormal polynomials.

Using the argument in [13], the Schur function associated to the measure $|\varphi(e^{i\theta})|^2 d\mu(e^{i\theta})$ is

$$g_k(z) = C_1 \frac{\alpha_k + C_2}{1 + \overline{\alpha}_k C_2}$$

where

$$\begin{aligned} C_1 &= f(z; -\overline{\alpha}_{k-1}, -\overline{\alpha}_{k-1}, \ldots, -\overline{\alpha}_0, 1) \\ C_2 &= z f(z; \alpha_{k+1}, \alpha_{k+2}, \ldots) \end{aligned}$$

where f is the Schur function associated to the family of Verblunsky coefficients S (recalling Verblunsky's Theorem, which associates to every measure a sequence of Verblunsky coefficients). Then, using the reasoning in [13] we see that

$$\left|F_{kk}(z, \mathcal{C}_\alpha^{(n)})\right| \le \left| \frac{2}{1 - zC_1 \frac{\alpha_k + C_2}{1+\overline{\alpha}_k C_2}} \right|.$$

One important thing to note is that the constants z, C_1, C_2 do not depend on α_k and $|C_1|, |C_2| \le 1$ and $|z| < 1$. We will use the above identity to bound $\mathbb{E}(|F_{kk}(z, \mathcal{C}_\alpha^{(n)})|^s | \{\alpha_i\}_{i\neq k})$.

To do this we consider the following

$$\int_{\mathbb{D}(0,R)} \left| \frac{2}{1 - zC_1 \frac{\alpha_k + C_2}{1+\overline{\alpha}_k C_2}} \right|^s \frac{1}{|\alpha_k|^\sigma} d\lambda(\alpha_k).$$

By the same reasoning as in [13], it suffices to bound

$$\sup_{\omega_1, \omega_2 \in \mathbb{D}} \int_{\mathbb{D}(0,R)} \left| \frac{2}{1 - \omega_1 \frac{\alpha_k + \omega_2}{1+\overline{\alpha}_k \omega_2}} \right|^s \frac{1}{|\alpha_k|^\sigma} d\lambda(\alpha_k).$$

Clearly (by the same reasoning as in [13])

$$\left| \frac{2}{1 - \omega_1 \frac{\alpha_k + \omega_2}{1+\overline{\alpha}_k \omega_2}} \right| \le \left| \frac{4}{1 + \overline{\alpha}_k \omega_2 - \omega_1(\alpha_k + \omega_2)} \right|.$$

For $\alpha_k = x + iy$ and the same reasoning as in [13], we have $1 + \overline{\alpha}_k\omega_2 - \omega_1(\alpha_k + \omega_2) = x(-\omega_1+\omega_2)+y(-i\omega_1-i\omega_2)+(1-\omega_1\omega_2)$ where not all of $(-\omega_1+\omega_2), (-i\omega_1-i\omega_2), (1-\omega_1\omega_2)$ can be small.

If $|(-\omega_1 + \omega_2)| \geq \epsilon$ then

$$\begin{aligned}
\int_{\mathbb{D}(0,R)} \left| \frac{2}{1 - \omega_1 \frac{\alpha_k + \omega_2}{1+\overline{\alpha}_k\omega_2}} \right|^s \frac{1}{|\alpha_k|^\sigma} d\lambda(\alpha_k) &\leq \frac{4^s}{\epsilon^s} \int_{-R}^{R} \int_{-R}^{R} \frac{1}{|x + yD + E|^s} \frac{1}{|x^2 + y^2|^{\sigma/2}} dxdy \\
&\leq \frac{4^s}{\epsilon^s} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{|x + yD + E|^s} \frac{1}{|x^2 + y^2|^{\sigma/2}} dxdy \\
&\leq \frac{4^s}{\epsilon^s} \int_{-1}^{1} \int_{-1}^{1} \frac{1}{|x|^s} \frac{1}{|x^2 + y^2|^{\sigma/2}} dxdy \\
&\leq \frac{4^s}{\epsilon^s} \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \frac{1}{|\cos(\theta)|^s} \frac{r}{|r|^{s+\sigma}} drd\theta \\
&= \frac{4^s}{\epsilon^s} \frac{(\sqrt{2})^{2-s-\sigma}}{2-s-\sigma} \int_{0}^{2\pi} \frac{1}{|\cos(\theta)|^s} d\theta \\
&= \frac{4^s}{\epsilon^s} \frac{(\sqrt{2})^{2-s-\sigma}}{2-s-\sigma} \left(4 \int_{0}^{\pi/2} \csc(\theta)^s d\theta \right) \\
&< \frac{4^{s+1}}{\epsilon^s} \frac{(\sqrt{2})^{2-s-\sigma}}{2-s-\sigma} C_s,
\end{aligned}$$

where we used Lemma 3.1.3 to obtain the constant C_s. We attain the same bound for $|(\omega_1 + \omega_2)| \geq \epsilon$ (replacing y with x when we invoke Corollary 3.1.9).

If $|(-\omega_1 + \omega_2)| \leq \epsilon$ and $|(\omega_1 + \omega_2)| \leq \epsilon$ then

$$|x(-\omega_1 + \omega_2) + y(-i\omega_1 - i\omega_2) + (1 - \omega_1\omega_2)| \geq (1 - \epsilon^2 - 4\epsilon)$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{D}(0,R)} \left| \frac{2}{1 - \omega_1 \frac{\alpha_k + \omega_2}{1+\overline{\alpha}_k\omega_2}} \right|^s \frac{1}{|\alpha_k|^\sigma} d\lambda(\alpha_k) \\
\leq\ & 2^{s+2} \left(\frac{1}{1 - \epsilon^2 - 4\epsilon} \right)^s \int_{-1}^{1} \int_{-1}^{1} \frac{1}{|x^2 + y^2|^{\sigma/2}} dxdy \\
\leq\ & 2^{s+2} \left(\frac{1}{1 - \epsilon^2 - 4\epsilon} \right)^s \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} drd\theta \\
=\ & 2^{s+3}\pi\sqrt{2} \left(\frac{1}{1 - \epsilon^2 - 4\epsilon} \right)^s.
\end{aligned}$$

Therefore, we get the desired result with

$$C = \max\left\{ \frac{4^{s+1}}{\epsilon^s} \frac{(\sqrt{2})^{2-s-\sigma}}{2-s-\sigma} C_s, 2^{s+3}\pi\sqrt{2} \left(\frac{1}{1-\epsilon^2-4\epsilon}\right)^s \right\}$$

completing the proof. □

It is interesting to note that the bound derived above is the reason we insist that $0 < \sigma \leq 1$ since we want to apply the lemma to any $s \in (0,1)$.

In [13], Stoiciu shows that the above lemmas are sufficient to prove the desired result of obtaining Aizenman-Molchanov bounds for the fractional moments of the resolvent of CMV matrices. One key step in the proof of this theorem is to apply the resolvent identity twice using two different decoupled forms of the matrix $\mathcal{C}^{(n)}$. The end result is that for fixed k, there exists an $m \geq 0$ such that for any s_1 satisfying $|s_1 - (k+m)| \leq 2$ we can find constants C and β independent of k with $\beta < 1$ such that

$$\mathbb{E}\Big(\big|(\mathcal{C}_1 - z)^{-1}_{ks_1}\big|^s\Big) \leq C\beta$$

where $\mathcal{C}_1$ is a decoupled CMV matrix. To find a bound for $\mathbb{E}(|(\mathcal{C} - z)^{-1}_{kl}|^s)$ we can repeat this process $\left[\frac{l-k}{m+3}\right]$ times, each time moving $(m+3)$ spots to the right from k to l to obtain

$$\mathbb{E}\Big(\big|(\mathcal{C} - z)^{-1}_{kl}\big|^s\Big) \leq C\beta^{(l-k)/(m+3)},$$

which immediately gives the desired Aizenman-Molchanov bounds.

3.2 The Road Map

Now that we have established our first desired result, we can proceed with the proof of Step 2: Exponential localization of the eigenfunctions. By the same reasoning as in [13], we can use the results of Step 1 and apply Aizenman's Theorem for CMV matrices to conclude that there exist positive constants C_0 and D_0 depending on s such that

$$\mathbb{E}(\sup_{j\in\mathbb{Z}} |(\delta_k, (\mathcal{C}^{(n)}_\alpha)^j \delta_l)|) \leq C_0 e^{-D_0|k-l|}.$$

Using this result and the methods used in [13], we have the following lemma.

Lemma 3.2.1. [13] *For almost every $\alpha \in \Omega$ there exists a constant $C_\alpha > 0$ such that for any n and any k, l satisfying $1 \leq k, l, \leq n$ and $|k-l| \geq \frac{12}{D_0}\ln(n+1)$, we have*

$$\sup_{j\in\mathbb{Z}} |(\delta_k, (\mathcal{C}^{(n)}_\alpha)^j \delta_l)| \leq C_\alpha e^{\frac{-D_0}{2}|k-l|}.$$

From this, we can follow the procedure given in [13] and complete the proof of Step 2. Indeed we have the following theorem.

Theorem 3.2.2. [13] *There exists a constant $D_2 > 0$ and for almost every $\alpha \in \Omega$ there exists a constant $C_\alpha > 0$ such that for any unitary eigenfunction $\varphi^{(n)}_\alpha$ of $\mathcal{C}^{(n)}_\alpha$, there exists a point $m(\varphi^{(n)}_\alpha)$ with $1 \leq m(\varphi^{(n)}_\alpha) \leq n$ such that for any $m \in \mathbb{Z}$ satisfying $|m - m(\varphi^{(n)}_\alpha)| \geq D_2 \ln(n+1)$, we have*

$$|\varphi^{(n)}_\alpha(m)| \leq C_\alpha e^{-(4/D_2)|m-m(\varphi^{(n)}_\alpha)|}$$

where we call the point $m(\varphi^{(n)}_\alpha)$ the center of localization.

Outline of Proof. Let $e^{i\theta_\alpha}$ be an eigenvalue of $\mathcal{C}_\alpha^{(n)}$ and let $\varphi_\alpha^{(n)}$ be the corresponding eigenfunction. The sequence of functions defined by

$$f_M(e^{i\theta}) = \frac{1}{2M+1} \sum_{j=-M}^{M} e^{ij(\theta-\theta_\alpha)}$$

is uniformly bounded by 1 and converges pointwise to the characteristic function of the point $e^{i\theta_\alpha}$. By applying Lemma 3.2.1 and the functional calculus to $\mathcal{C}_\alpha^{(n)}$ we gett that

$$|(\delta_k, f_M(\mathcal{C}_\alpha^{(n)})\delta_l)| \leq C_\alpha e^{\frac{-D_0}{2}|k-l|}.$$

It follows that

$$|\varphi_\alpha^{(n)}(k)\varphi_\alpha^{(n)}(l)| \leq C_\alpha e^{\frac{-D_0}{2}|k-l|}.$$

Now we can choose as center of localization the smallest integer m such that

$$|\varphi_\alpha^{(n)}(m(\varphi_\alpha^{(n)}))| = \max_m |\varphi_\alpha^{(n)}(m)|$$

and derive the desired conclusion.

Now that we have completed the proofs of Steps 1 and 2, we can proceed with the proof of the next step required to prove the existence of Poisson statistics in the asymptotic distribution of the eigenvalues. To do so, we must decouple the matrix into smaller matrices and use these smaller matrices to approximate the behavior of the original matrix. We begin by considering the matrix $\tilde{\mathcal{C}}^{(n)}$ obtained in the same way as $\mathcal{C}^{(n)}$ with the additional restriction that $\alpha_{[\frac{n}{\ln(n)}]} = e^{i\eta_1}, \alpha_{2[\frac{n}{\ln(n)}]} = e^{i\eta_2}, \ldots, \alpha_{n-1} = e^{i\eta_{[\ln(n)]}}$ (where $[x]$ denotes the greatest integer less than or equal to x). Due to this additional restriction, the matrix $\mathcal{C}^{(n)}$ decouples into approximately $[\ln(n)]$ smaller matrices.

The main idea behind the proof is that this additional restriction that we have introduced to define the matrix $\tilde{\mathcal{C}}^{(n)}$ creates a negligible error when trying to approximate the point process define by $\mathcal{C}^{(n)}$ because the center of localization for the eigenfunctions of both matrices usually avoids the regions at which the matrix $\tilde{\mathcal{C}}^{(n)}$ decouples. Using the methods in [13], we can make a rigorous statement about what we mean by "usually". Therefore, when we replace $\mathcal{C}_\alpha^{(N)}$ with $\tilde{\mathcal{C}}^{(N)}$, our estimates concerning the point process acquire an error that becomes negligible as $N \to \infty$, which is what we wanted to show.

Finally, we can proceed with the proof of the existence of Poisson statistics. The previous results show that we can estimate the point process determined by a matrix $\mathcal{C}^{(N)}$ by considering the point processes of $[\ln(n)]$ matrices of size $[\frac{n}{\ln(n)}]$ (denoted $\mathcal{C}_1^{(N)}, \ldots \mathcal{C}_{[\ln(n)]}^{(N)}$). Given an interval $I_N = (e^{i(\theta_0+\frac{2\pi a}{N})}, e^{i(\theta_0+\frac{2\pi b}{N})}$, we can show that each of the matrices $\mathcal{C}_1^{(N)}, \ldots \mathcal{C}_{[\ln(n)]}^{(N)}$ has at most one eigenvalue in the interval I_N up to a negligible error. Here we provide an outline of the steps taken in [13] to give this result.

If we let $A(m, \mathcal{C}, I)$ be the event "$\mathcal{C}$ has at least m eigenvalues in the interval I" and let $M(e^{i\theta})$ be the event "$e^{i\theta}$ is an eigenvalue of $\mathcal{C}$" then we have the following lemma.

Lemma 3.2.3. [13] *With $\mathcal{C}^{(N)}$ and I_N defined as before and for any $e^{i\theta_1} \in I_N$ we have*

$$\mathbb{P}(A(2, \mathcal{C}^{(N)}, I_N | M(e^{i\theta_1})) \leq (b-a).$$

Using this result, we can make a very strong statement about the probability of having 2 or more eigenvalues in I_N. To this effect we have the following theorem.

Theorem 3.2.4. [13] *Using the notation defined before, we have*

$$\mathbb{P}(A(2, \mathcal{C}^{(N)}, I_N)) \leq \frac{(b-a)^2}{2}.$$

Using this result, it follows (as is shown in [13]) that $\mathbb{P}(A(2, \mathcal{C}_k^{(N)}, I_N)) = O([\ln(n)]^{-2})$ as $n \to \infty$ for all k satisfying $1 \leq k \leq [\ln(n)]$. Therefore, the probability that the direct sum of all of the matrices $\mathcal{C}_k^{(N)}$ has two or more eigenvalues in the interval I_N is $[\ln(n)]O([\ln(n)]^{-2})$ and therefore goes to zero as $n \to \infty$. The importance of this result for us cannot be overstated. It allows us to model the occurrence of an eigenvalue of the matrix $\tilde{\mathcal{C}}^{(N)}$ in the interval I_N as a Bernoulli trial and we can therefore conclude that the asymptotic distribution of the eigenvalues of $\tilde{\mathcal{C}}^{(N)}$ is almost surely Poisson. Finally, using our results from before, we know that the asymptotic distribution of the eigenvalues of $\tilde{\mathcal{C}}^{(N)}$ is the same as that of $\mathcal{C}^{(N)}$ and we have achieved the desired result.

Chapter 4

Joye Matrices

4.1 Joye's Result

In Section 1.3, we mentioned the work of Blatter and Browne in [1] involving a unitary band matrix with random phases to describe the time evolution of the energy state of an electron in a ring threaded by a time dependent magnetic flux. In [3], Joye proved several meaningful results about classes of such operators, including conditions under which we almost surely have Aizenman-Molchanov bounds and pure point spectrum. He studies these matrices in the following way.

Define a probability space Ω and a map θ_k by

$$\theta_k : \Omega \to \mathbb{T}, \quad \text{s.t.} \quad \theta_k(\omega) = \omega_k, \quad k \in \mathbb{Z}$$

where $\mathbb{T} = \mathbb{R} \bmod 2\pi$. With this notation we can define a unitary operator U_ω by

$$U_\omega = D_\omega S_0 \quad \text{with} \quad D_\omega = \mathrm{diag}\{e^{-i\theta_k(\omega)}\}$$

and

$$S_0 = \begin{pmatrix} \ddots & rt & -t^2 & & & \\ & r^2 & -rt & & & \\ & rt & r^2 & rt & -t^2 & \\ & -t^2 & -rt & r^2 & -rt & \\ & & & rt & r^2 & \\ & & & -t^2 & -rt & \ddots \end{pmatrix}$$

where the r^2 entries are along the diagonal (this construction comes from [3]).

For $z \in \mathbb{D}$ define

$$H_\omega(z) = U_\omega(U_\omega - z)^{-1}.$$

In [3], Joye proved the following result.

Theorem 4.1.1. [3] *Let U_ω be defined as above and suppose that $\{\theta_k(\omega)\}_{k\in\mathbb{Z}}$ is a collection of i.i.d. random variables and are distributed according to the probability measure $d\nu(\theta) = \tau(\theta)d\theta$, where $\tau(\theta) \in \mathcal{L}^\infty(\mathbb{T})$. Let $s \in (0,1)$. There exists a $t_0(s) > 0$*

small enough and $0 < K(s) < \infty$ *such that if* $|t| < t_0(s)$*, there exists* $\gamma(s,t) > 0$ *so that for any* $j, k \in \mathbb{Z}$ *we have*

$$\mathbb{E}(|H_\omega(z)_{jk}|^s) \leq K(s) e^{-\gamma(s,t)|j-k|}.$$

(There actually exists a stronger version of this theorem presented in [4], but we choose to work with this version because the proof can easily be generalized in a way that will eventually allow us to decouple this matrix as described in Chapter 2.)

Clearly this result gives fractional moment estimates on the resolvent of the matrix U_ω. In [3], Joye goes on to show that this implies that the spectrum of U_ω is almost surely pure point as follows. If we let δ_0 be a cyclic vector associated to H_ω then by the Spectral Theorem, there exists a measure μ_ω such that

$$\langle \delta_0 | H_\omega(z) \delta_0 \rangle = \int_{\partial\mathbb{D}} \frac{e^{i\theta}}{e^{i\theta} - z} d\mu_\omega(\theta) \equiv J_\omega(z).$$

Now, given a nontrivial probability measure μ on $\partial\mathbb{D}$, let us define a function $G(z)$ as in [10] by

$$G(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\theta)}{|e^{i\theta} - z|^2}.$$

A simple computation shows that if $z \in \mathbb{D}$, we can write this as

$$G(e^{i\gamma}) = \int_{\partial\mathbb{D}} \frac{d\mu(\theta)}{4\sin^2(\frac{\theta-\gamma}{2})}.$$

If we recall that the Poisson integral of a measure μ is given by

$$P[d\mu](z) = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\mu(\theta) \geq 0, \quad |z| < 1,$$

then for $r < 1$ we can write

$$\begin{aligned} G(re^{i\gamma}) &= P[d\mu_\omega](re^{i\gamma}) + \int_{\partial\mathbb{D}} \frac{r^2}{1 + r^2 - 2r\cos(\theta - \gamma)} d\mu_\omega(\theta) \\ &\equiv P[d\mu_\omega](re^{i\gamma}) + B_\omega(r, \gamma). \end{aligned}$$

An application of the Monotone Convergence Theorem shows that

$$B_\omega(\gamma) \equiv \lim_{r \uparrow 1} B_\omega(r, \gamma) = G(e^{i\gamma}).$$

In [3], Joye uses Aizenmann-Molchanov bounds on the fractional moments to show that $B_\omega(\gamma)$ is finite for a.e. γ with respect to Lebesgue measure.

Now let us define a perturbation of the matrix U_ω by perturbing the matrix D_ω. Let us define $\hat{D}_\omega$ to be identical to D_ω except that the $(0, 0)$ entry has been changed to a 1 and let us define $\hat{H}_\omega$ accordingly. Then we have the relation

$$J_\omega(z) = \frac{\hat{J}_\omega(z)}{\hat{J}_\omega(z)(1 - e^{i\theta_0(\omega)}) + e^{i\theta_0(\omega)}}. \tag{4.1}$$

Although it is natural to think of $\hat{U}_\omega$ as a rank one perturbation of U_ω, we can also think of U_ω as a rank one perturbation of $\hat{U}_\omega$. We will in fact use this method of thought to apply the results of [2]. In that paper, Combescure proves the following theorem, which is essential for our purposes.

Theorem 4.1.2. *(Simon-Wolff Criterion)(*[10]; [2]*) The following are equivalent:*

1. *For a.e. $\theta_0(\omega)$, the matrix U_ω has only pure point spectrum,*

2. *For a.e. $\theta \in \mathbb{T}$, $\hat{B}_\omega(\theta) < \infty$.*

From this theorem, we can see that to understand the measure μ_ω, it is essential that we discover the properties of $\hat{\mu}_\omega$. Since the relationship between $J_\omega(z)$ and $\hat{J}_\omega(z)$ is well understood, we can use the reasoning of [3] to conclude that $\hat{B}_\omega(\theta) < \infty$ for a.e. $\theta \in \mathbb{T}$. Therefore, by the above theorem, we can conclude that the matrix U_ω has only pure point spectrum as long as $\omega \in \Omega_0$ where $\Omega_0 \subset \Omega$ is a set of probability one. If we apply the same reasoning using the vector δ_j, then we get the same result except we replace "almost all values of $\theta_0(\omega)$" with "almost all values of $\theta_j(\omega)$," or analogously, we get the desired result as long as $\omega \in \Omega_j$ where $\Omega_j \subset \Omega$ is a set of probability one. Therefore, if $\omega \in \bigcap_{j\in\mathbb{Z}} \Omega_j$, we get the desired result, but $\bigcap_{j\in\mathbb{Z}} \Omega_j$ is a set of probability one, so we get the desired result almost always.

4.2 The Carathéodory Function

In [9], Simon shows that if we are given a Carathéodory function F on $\mathbb{D}$ with Taylor expansion at $z = 0$ given by

$$F(z) = 1 + 2\sum_{n=1}^{\infty} c_n z^n$$

then

$$F(z) = \int_{\partial\mathbb{D}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)$$

where μ is a measure on $\partial\mathbb{D}$ satisfying

$$\int_{\partial\mathbb{D}} e^{-in\theta} d\mu(\theta) = c_n.$$

He also states that the measure of an isolated point $e^{i\gamma}$ is given by

$$\mu(e^{i\gamma}) = \lim_{r\uparrow 1} \left(\frac{1-r}{2}\right) F(re^{i\gamma})$$

and so $e^{i\gamma}$ is a pure point of the measure μ if and only if this limit is nonzero. Similarly, in [3], Joye defines a similar function as follows. Given a measure μ on $\partial\mathbb{D}$, define the *Joye function*, $J(z)$, by

$$J(z) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}}{e^{i\theta} - z} d\mu(\theta).$$

By the same reasoning as above we have

$$\mu(e^{i\gamma}) = \lim_{r\uparrow 1}(1-r)J(re^{i\gamma}).$$

To understand the relationship between these two functions, we have the following theorem.

Theorem 4.2.1. *Given a probability measure μ on $\partial\mathbb{D}$, define the functions F and J as above. Then for any $z \in \mathbb{D}$ we have*

$$2J(z) = F(z) + 1.$$

This theorem is not a tremendously surprising result. Given the above formulae for $\mu(e^{i\gamma})$, we would expect that $2J(z) \sim F(z)$ if these quantities were to become very large.

Proof. Clearly we have

$$F(re^{i\gamma}) = J(re^{i\gamma}) + \int_{\partial\mathbb{D}} \frac{re^{i\gamma}}{e^{i\theta} - re^{i\gamma}} d\mu(\theta).$$

Let us examine this last term. We have

$$\begin{aligned}
&\int_{\partial\mathbb{D}} \frac{re^{i\gamma}}{e^{i\theta} - re^{i\gamma}} d\mu(\theta) \\
= \quad & re^{i\gamma} \int_{\partial\mathbb{D}} e^{-i\theta} \frac{1}{1 - re^{i(\gamma-\theta)}} d\mu(\theta) \\
= \quad & re^{i\gamma} \int_{\partial\mathbb{D}} e^{-i\theta}(1 + re^{i(\gamma-\theta)} + (re^{i(\gamma-\theta)})^2 + (re^{i(\gamma-\theta)})^3 + \cdots) d\mu(\theta) \\
= \quad & re^{i\gamma} \int_{\partial\mathbb{D}} (e^{-i\theta} + re^{i(\gamma-2\theta)} + r^2 e^{i(2\gamma-3\theta)} + r^3 e^{i(3\gamma-4\theta)} + \cdots) d\mu(\theta) \\
= \quad & \int_{\partial\mathbb{D}} (re^{i(\gamma-\theta)} + r^2 e^{i(2\gamma-2\theta)} + r^3 e^{i(3\gamma-3\theta)} + r^4 e^{i(4\gamma-4\theta)} + \cdots) d\mu(\theta) \\
= \quad & \int_{\partial\mathbb{D}} \left(\sum_{n=1}^{\infty} r^n e^{in(\gamma-\theta)} \right) d\mu(\theta) \\
= \quad & \left(\sum_{n=1}^{\infty} r^n e^{in\gamma} \int_{\partial\mathbb{D}} e^{-in\theta} d\mu(\theta) \right) \\
= \quad & \sum_{n=1}^{\infty} c_n (re^{i\gamma})^n \\
= \quad & \frac{F(re^{i\gamma}) - 1}{2}
\end{aligned}$$

where we used the absolute convergence of the series to justify integrating it term by term. Therefore,

$$F(re^{i\gamma}) = J(re^{i\gamma}) + \frac{F(re^{i\gamma}) - 1}{2}$$

and we conclude that for $0 \leq r < 1$

$$F(re^{i\gamma}) + 1 = 2J(re^{i\gamma})$$

as desired. □

4.3 Alexandrov Measures

In [9], Simon introduces a particular family of perturbed measures called "*Alexandrov Measures*" that will be of particular importance for us. Before we can give a formal description, we need the following definition.

Definition 2. *If F is a Carathéodory function and Q is analytic from $\mathbb{C}_r = \{z|\mathrm{Re}(z) > 0\}$ to itself with $Q(1) = 1$ then $Q(F(z))$ is also a Carathéodory function.*

Let us define functions L, R

$$L(z) = \frac{1+z}{1-z}$$
$$L^{-1}(z) = \frac{z-1}{z+1}$$
$$R_\lambda(z) = \lambda z, \quad \lambda \in \partial\mathbb{D}.$$

The function L conformally maps $\mathbb{C}_r$ to $\mathbb{D}$ with 1 to 0 and R_λ maps $\mathbb{D}$ to itself with $R_\lambda(0) = 0$. Then $Q_\lambda = L \circ R_\lambda \circ L^{-1}$ is an analytic function from $\mathbb{C}_r$ to itself. It is easily computed that

$$Q_\lambda(z) = \frac{(1-\lambda) + (1+\lambda)z}{(1+\lambda) + (1-\lambda)z}.$$

Therefore, given a Carathéodory function F,

$$F^{(\lambda)}(z) = \frac{(1-\lambda) + (1+\lambda)F(z)}{(1+\lambda) + (1-\lambda)F(z)}$$

is also a Carathéodory function for any $\lambda \in \partial\mathbb{D}$ and we will call the associated family of measures Alexandrov measures and denote them by $d\mu_\lambda(\theta)$.

4.4 Singular Spectrum

In this section, we will state and prove a result that is essential to understanding the Aronszajn-Donaghue Theory discussed in Section 4.5. We will prove that the support of the singular part of a measure μ_s on $\partial\mathbb{D}$ is contained in the set of all $e^{i\theta}$ such that $\lim_{r\uparrow 1} P[d\mu_s](re^{i\theta}) = \infty$. This is not a new theorem, but it is presented here with an original proof. Simon, in [9], cites page 243 in [8] for this result. This may be a typo. The relevant lemma we will use here is Theorem 7.15 in [8] found on page 143 and says the following.

Theorem 4.4.1. *Let μ be a positive Borel measure on $\mathbb{R}$. Define $D\mu(x)$ by*

$$D\mu(x) = \lim_{n\to\infty} \frac{m(I_n)}{\mu(I_n)}$$

where m denotes Lebesgue measure and $\{I_n\}_{n\geq 0}$ is a sequence of intervals centered at x satisfying $\cap_{n\geq 0} I_n = \{x\}$. If $\mu \perp m$ then

$$D\mu(x) = \infty$$

for μ-a.e. x.

We will assume that this theorem applies replacing $\mathbb{R}$ with $\mathbb{T}$ and replacing m with σ (where $\sigma = \frac{m}{2\pi}$ is a normalized version of m). Notice that this theorem applies to a.e. x with respect to the measure μ. It is obvious that $D\mu(x) = \infty$ if x is a pure point of the measure μ. The above theorem says that, in some sense, the support of a singular measure can in some ways be treated as pure points. Now we can proceed with our main result, but first we need the following lemma.

Lemma 4.4.2. *Given any integer $n \geq 4$, there exists $r_n < r'_n < 1$ such that for all $r \in (r_n, r'_n)$ we have*

$$\frac{1-r^2}{(1+r^2-2r\cos(\frac{1}{n}))^2} > \pi n.$$

Furthermore,

$$\lim_{n\to\infty} r'_n = 1.$$

Proof. Consider the equation

$$1-r^2 = \pi n(1+r^2-2r\cos(1/n))^2.$$

We see that when $r = 0$, the left side is smaller than the right side. If we take the limit as $r \longrightarrow 1$, again we see that the left side is smaller than the right side. Thus, if there exists some $r \in (0,1)$ such that the left side is larger than the right side, then there would have to be an interval $(r_n, r'_n) \subset (0,1)$ such that for all $r \in (r_n, r'_n)$, the left side is larger than the right side. It is easily checked that if $r = \cos(1/n)$ then

$$\begin{aligned} & 1-r^2-\pi n(1+r^2-2r\cos(1/n))^2 \\ = \; & 1-\cos^2(1/n)-\pi n(1+\cos^2(1/n)-2\cos^2(1/n))^2 \\ = \; & \sin^2(1/n)-\pi n(\sin^2(1/n))^2 \\ = \; & \sin^2(1/n)-\pi n\sin^4(1/n) > 0 \end{aligned}$$

as long as $n \geq 4$. Furthermore, since

$$\lim_{n\to\infty} \cos(1/n) = 1$$

it must be the case that

$$\lim_{n\to\infty} r'_n = 1$$

as desired. □

Now we have our desired result.

Theorem 4.4.3. *If $e^{i\gamma}$ is in the support of the singular part of a measure μ on $\partial\mathbb{D}$ then*

$$\lim_{r\uparrow 1} P[d\mu_s](e^{i\gamma}) = \infty$$

where

$$P[d\mu_s](re^{i\gamma}) = \int_{\partial\mathbb{D}} \frac{1-r^2}{|e^{i\theta}-re^{i\gamma}|^2} d\mu_s(\theta).$$

Proof. Let $e^{i\gamma} \in \text{supp}(\mu_s)$. For a given value of $r < 1$, define the intervals I_n of the form

$$I_n = (e^{i(\gamma-\frac{1}{n})}, e^{i(\gamma+\frac{1}{n})}).$$

Define y_n by

$$y_n = \min_{z\in I_n}\{|z - re^{i\gamma}|^{-2}\}.$$

Then

$$\begin{aligned} P[d\mu_s](re^{i\gamma}) &= \int_{\partial\mathbb{D}} \frac{1-r^2}{|e^{i\theta}-re^{i\gamma}|^2} d\mu_s(\theta) \\ &\geq \int_{I_n} \frac{1-r^2}{|e^{i\theta}-re^{i\gamma}|^2} d\mu_s(\theta) \\ &\geq \int_{I_n} y_n(1-r^2) d\mu_s(\theta) \end{aligned}$$

where we will choose n later.

Clearly y_n will be the distance from $re^{i\gamma}$ to the endpoints of the interval I_n. Therefore,

$$\begin{aligned} y_n &= |e^{i(\gamma-\frac{1}{n})} - re^{i\gamma}|^{-2} \\ &= \frac{1}{(1+r^2-2r\cos(\frac{1}{n}))^2} \end{aligned}$$

and $\sigma(I_n) = \frac{1}{2\pi}(\frac{2}{n}) = \frac{1}{\pi n}$. It follows then that the following are equivalent

$$\begin{aligned} & (1-r^2)y_n > \frac{1}{\sigma(I_n)} \\ \iff & \frac{1-r^2}{(1+r^2-2r\cos(\frac{1}{n}))^2} > \pi n \\ \iff & r \in (r_n, r_n') \end{aligned}$$

by Lemma 4.4.2. Then, by what we derived before, we have

$$\begin{aligned} P[d\mu_s](re^{i\gamma}) &\geq \int_{I_n} y_n(1-r^2) d\mu_s(\theta) \\ &> \int_{I_n} \frac{1}{\sigma(I_n)} d\mu_s(\theta) \\ &= \frac{\mu_s(I_n)}{\sigma(I_n)} \end{aligned}$$

for every n such that $r \in (r_n, r'_n)$. As r gets closer to 1, this is true for larger n, so the conclusion follows from Theorem 4.4.1. □

4.5 Aronszajn-Donaghue Theory

Many of the results of this section are taken from [10], though an equivalent form of the result is given in [3] and the result of Section 4.2 will help explain the connection. The Aronszajn-Donaghue Theory is a very thorough description of the supports of the pure point, singular continuous, and absolutely continuous parts of a measure. The first result we need is the following theorem taken from [10].

Theorem 4.5.1. [10] *Given a nontrivial probability measure μ on $\partial\mathbb{D}$, we have the following:*

1. $\lim_{r\uparrow 1} \frac{\mathrm{Re}(F(re^{i\gamma}))}{1-r} = 2G(e^{i\gamma})$

2. *If $G(e^{i\gamma}) < \infty$, then $F(e^{i\gamma})$ exists and is pure imaginary.*

3. *If $G(e^{i\gamma}) < \infty$, then $\lim_{r\uparrow 1} \frac{F(e^{i\gamma})-F(re^{i\gamma})}{1-r} = -2G(e^{i\gamma})$.*

We will adapt the proof presented it [10] to achieve a similar result for the *Joye function.*

Theorem 4.5.2. *Given a nontrivial probability measure μ on $\partial\mathbb{D}$, we have the following:*

1. $\lim_{r\uparrow 1} \frac{2\mathrm{Re}(J(re^{i\gamma}))-1}{1-r} = 2G(e^{i\gamma})$

2. *If $G(e^{i\gamma}) < \infty$, then $J(e^{i\gamma})$ exists and is of the form $\frac{1}{2} + iy$ with $y \in \mathbb{R}$.*

3. *If $G(e^{i\gamma}) < \infty$, then $\lim_{r\uparrow 1} \frac{F(e^{i\gamma})-F(re^{i\gamma})}{1-r} = -G(e^{i\gamma})$.*

Proof. For part 1 we use the relation given in Theorem 4.2.1 and the result of Theorem 4.5.1 part (i). We have

$$\begin{aligned} 2G(e^{i\gamma}) &= \lim_{r\uparrow 1} \frac{\mathrm{Re}(F(re^{i\gamma}))}{1-r} \\ &= \lim_{r\uparrow 1} \frac{2\mathrm{Re}J(re^{i\gamma}))-1}{1-r} \end{aligned}$$

as desired.

For part 2, we note that if $G(e^{i\gamma})$ is finite, then part 1 shows that

$$2\mathrm{Re}J(e^{i\gamma}))-1=0$$

so $\mathrm{Re}J(e^{i\gamma})) = \frac{1}{2}$. Similarly, using the results of Theorem 4.5.1, we see that $F(e^{i\gamma})$ exists and is pure imaginary. Since $J(e^{i\gamma}) = \frac{1}{2}(F(e^{i\gamma})+1)$ we get that $J(e^{i\gamma})$ exists and has real part equal to $\frac{1}{2}$ as desired.

For part 3, we have

$$\frac{\partial}{\partial r}\left(\frac{e^{i\theta}}{e^{i\theta}-re^{i\gamma}}\right)=\frac{e^{i(\theta-\gamma)}}{(e^{i(\theta-\gamma)}-r)^2}\to\frac{-1}{|e^{i\theta}-e^{i\gamma}|^2}$$

as $r\uparrow 1$. Thus, the Dominated Convergence Theorem shows that if $G(e^{i\gamma})$ is finite then

$$\lim_{r\uparrow 1}\frac{\partial}{\partial r}J(re^{i\gamma})\to -G(e^{i\gamma}).$$

This implies

$$J(e^{i\gamma})-J(re^{i\gamma})=\int_r^1\frac{\partial}{\partial r}J(\rho e^{i\gamma})d\rho$$

and this in turn implies the desired relation. □

Using Theorem 4.5.1 we have the following.

Theorem 4.5.3. [10] *Given a nontrivial probability measure μ on $\partial\mathbb{D}$, if $e^{i\gamma}$ is a pure point of μ_λ for $\lambda\neq 1$, then*

$$G(e^{i\gamma})<\infty,\quad \textit{and}\quad F(e^{i\gamma})=\frac{\lambda+1}{\lambda-1}.$$

Conversely, if $G(e^{i\gamma})<\infty$ and λ is defined by $F(e^{i\gamma})$ as above then $e^{i\gamma}$ is a pure point of μ_λ.

Now we will again use the reasoning of [3] backwards and consider U_ω as a perturbation of $\hat{U}_\omega$. Recall that in order to get $\hat{U}_\omega$ from U_ω, we changed the $(0,0)$ entry of the matrix D_ω from $e^{i\theta_0(\omega)}$ to 1. Therefore, to get back to U_ω, we simply undo this change. Using the reasoning of [3], we get

$$\hat{J}_\omega(z)=\frac{J_\omega(z)(e^{i\theta_0})}{(e^{i\theta_0}-1)J_\omega(z)+1}$$

or equivalently

$$\hat{J}_\omega(z)=\frac{J_\omega(z)}{(1-e^{-i\theta_0})J_\omega(z)+e^{-i\theta_0}}.$$

We see that this is the same relation derived in Section 4.1 with J_ω and $\hat{J}_\omega$ switched and $e^{-i\theta_0}$ replacing $e^{i\theta_0}$. Using Theorem 4.5.2 and a similar argument to the proof of Theorem 10.1.3 in [10], we get the following.

Theorem 4.5.4. *Given a nontrivial probability measure μ_ω on $\partial\mathbb{D}$, if $e^{i\gamma}$ is a pure point of $\hat{\mu}_\omega$ (where we perturb the measure as in Section 4.1, Equation 4.1), then*

$$G(e^{i\gamma})<\infty,\quad \textit{and}\quad J(e^{i\gamma})=\frac{e^{-i\theta_0}}{e^{-i\theta_0}-1}.$$

Conversely, if $G(e^{i\gamma})<\infty$ and $e^{-i\theta_0}$ is defined by $J(e^{i\gamma})$ as above then $e^{i\gamma}$ is a pure point of $\hat{\mu}_\omega$.

Proof. Suppose $e^{i\gamma}$ is a pure point of $\hat{\mu}_\omega$. By what we said in Section 4.2, we have that

$$\hat{\mu}_\omega(e^{i\gamma}) = \lim_{r\uparrow 1}(1-r)\hat{J}_\omega(re^{i\gamma}).$$

Clearly $\lim_{r\uparrow 1}(1-r) = 0$ so if the above limit is nonzero, it must be that $\lim_{r\uparrow 1}\hat{J}_\omega(re^{i\gamma}) = \infty$. Recall also that

$$\hat{J}_\omega(z) = \frac{J_\omega(z)}{(1-e^{-i\theta_0})J_\omega(z) + e^{-i\theta_0}}.$$

From this it follows that

$$J(e^{i\gamma}) \equiv \lim_{r\uparrow 1} J(re^{i\gamma}) = \frac{e^{-i\theta_0}}{e^{-i\theta_0}-1}.$$

Using this, we can write

$$\begin{aligned}
\lim_{r\uparrow 1}(1-r)\hat{J}_\omega(re^{i\gamma}) &= \lim_{r\uparrow 1}(1-r)\frac{J_\omega(re^{i\gamma})}{(1-e^{-i\theta_0})J_\omega(re^{i\gamma}) + e^{-i\theta_0}} \\
&= \lim_{r\uparrow 1}(1-r)\frac{\frac{e^{-i\theta_0}}{e^{-i\theta_0}-1}}{(1-e^{-i\theta_0})(J_\omega(re^{i\gamma}) - J_\omega(e^{i\theta_0}))} \\
&= \lim_{r\uparrow 1}(1-r)\frac{-e^{-i\theta_0}}{(1-e^{-i\theta_0})^2(J_\omega(re^{i\gamma}) - J_\omega(e^{i\gamma}))} \\
&= \frac{-e^{-i\theta_0}}{2(1-e^{-i\theta_0})^2 G(e^{i\gamma})}
\end{aligned}$$

using part (iii) of Theorem 4.5.2. Since this quantity is nonzero, we get that $G(e^{i\gamma}) < \infty$ as desired. The proof of the converse is similar. ☐

Now we can use all of our results to get the following.

Theorem 4.5.5. *Given a nontrivial probability measure μ and $e^{-i\theta_0} \in (\partial\mathbb{D} \setminus \{1\}$, define the perturbed measure $\hat{\mu}$ so that the relation*

$$\hat{J}(z) = \frac{J(z)}{(1-e^{-i\theta_0})J(z) + e^{-i\theta_0}}$$

holds for all $z \in \mathbb{D}$. Then $\hat{\mu}$ is the Alexandrov measure μ_λ with $\lambda = e^{-i\theta_0}$.

Proof. Notice that

$$\begin{aligned}
& \hat{J}(z) = \frac{J(z)}{(1-e^{-i\theta_0})J(z) + e^{-i\theta_0}} \\
\iff\quad & \hat{F}(z) + 1 = \frac{2(F(z)+1)}{(1-e^{-i\theta_0})(F(z)+1) + 2e^{-i\theta_0}} \\
\iff\quad & \hat{F}(z) = \frac{2F(z) + 2 - F(z) - 1 + e^{-i\theta_0}F(z) + e^{-i\theta_0} - 2e^{-i\theta_0})}{(1+e^{-i\theta_0}) + (1-e^{-i\theta_0})F(z)} \\
\iff\quad & \hat{F}(z) = \frac{(1-e^{-i\theta_0}) + (1+e^{-i\theta_0})F(z)}{(1+e^{-i\theta_0}) + (1-e^{-i\theta_0})F(z)}.
\end{aligned}$$

It follows that $\hat{F}(z)$ is the Carathéodory function corresponding to the perturbed measure μ_λ with $\lambda = e^{-i\theta_0}$. Since the correspondence between nontrivial measures on $\partial\mathbb{D}$ and Carathéodory functions is one to one, the result follows. □

The intuition behind this result is that the approach taken in [3] can in some ways be thought of as Simon's approach in [10] done backwards. Simon uses properties of a measure μ to get information about the spectrum of a family of perturbed measures. Joye uses information about the perturbed family to characterize the spectrum of the original measure.

4.6 Perturbations Considered by Combescure

In [2], Combescure studies the resolvent of a unitary operator U by studying the behavior of the function $C(z)$ defined on $\mathbb{D}$ by

$$C(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\theta)}{e^{i\theta} - z}$$

where μ is the spectral measure corresponding to the operator U and some cyclic vector. We would like to gain a better understanding of the spectral measure μ and would therefore like to apply the results from [10], all of which are given in terms of the Carathéodory function F associated to the measure μ. To help us understand the relationship between $F(z)$ and $C(z)$ we have the following proposition.

Proposition 4.6.1. *Given a nontrivial probability measure μ on $\partial\mathbb{D}$, define the functions $F(z)$ and $C(z)$ as above. With this definition we have that for all $z \in \mathbb{D}$,*

$$2zC(z) = F(z) - 1.$$

Proof. In the proof of Theorem 4.2.1, we showed that for $0 \leq r < 1$,

$$\int_{\partial\mathbb{D}} \frac{re^{i\gamma}}{e^{i\theta} - re^{i\gamma}} d\mu(\theta) = \frac{F(re^{i\gamma}) - 1}{2}$$

and it is clear that

$$(re^{i\gamma})C(re^{i\gamma}) = \int_{\partial\mathbb{D}} \frac{re^{i\gamma}}{e^{i\theta} - re^{i\gamma}} d\mu(\theta).$$

The identity follows immediately. □

In [9], Simon shows that if F is the Carathéodory function associated to the measure μ then

$$F(z) = 1 + 2\sum_{n=1}^{\infty} c_n z^n$$

where

$$c_n = \int_{\partial\mathbb{D}} e^{-in\theta} d\mu(\theta).$$

It follows then that

$$C(z) = \sum_{n=1}^{\infty} c_n z^{n-1}.$$

By similar reasoning to that presented in [9], it is easy to show that the measure of the point $e^{i\gamma}$ is given by

$$\mu(e^{i\gamma}) = \lim_{r\uparrow 1}(1-r)(re^{i\gamma})C(re^{i\gamma}).$$

Clearly then, $e^{i\gamma}$ is a pure point of the measure μ if and only if this quantity is nonzero.

As in the other results we have discussed, Combescure in [2] examines perturbed measures. Given a nontrivial probability measure μ on $\partial\mathbb{D}$ and a complex number $\alpha \in \partial\mathbb{D}$ with $\alpha \neq 1$, define the perturbed measure $\mu^{(\alpha)}$ so that for all $z \in \mathbb{D}$

$$C_\alpha(z) = \frac{C(z)}{\alpha + (\alpha - 1)zC(z)}$$

where

$$C_\alpha(z) = \int_{\partial\mathbb{D}} \frac{d\mu^{(\alpha)}(\theta)}{e^{i\theta} - z}$$

and $C(z)$ is as we have defined it previously. With this definition of $\mu^{(\alpha)}$, we can use the same reasoning as Theorem 4.5.3 to conclude that if $e^{i\gamma}$ is a pure point of $\mu^{(\alpha)}$ then

$$\lim_{r\uparrow 1}(re^{i\gamma})C(re^{i\gamma}) = \frac{\alpha}{1-\alpha}.$$

Immediately we see the difficulty with carrying over the results from [10] to the function $C(z)$. All of our formulae have an extra factor of z preceding the $C(z)$. However, proposition 4.6.1 will help us work around this difficulty. We can rewrite the above as

$$\lim_{r\uparrow 1} \frac{F(re^{i\gamma}) - 1}{2} = \frac{\alpha}{1-\alpha},$$

which is equivalent to

$$F(e^{i\gamma}) = \lim_{r\uparrow 1} F(re^{i\gamma}) = \frac{\overline{\alpha} + 1}{\overline{\alpha} - 1}.$$

Using Proposition 1 from [2], we see that if μ is any nontrivial probability measure on $\partial\mathbb{D}$ and $e^{i\gamma}$ is a pure point of $\mu^{(\alpha)}$ then $G(e^{i\gamma}) < \infty$ (where $G(e^{i\gamma})$ is calculated with respect to the unperturbed measure μ). Therefore, using the exact same reasoning as before, we have the following, more complete, version of Theorem 4.5.5.

Theorem 4.6.1. *Let μ be a nontrivial probability measure on $\partial\mathbb{D}$. Given $\alpha \in \partial\mathbb{D}$ with $\alpha \neq 1$, define the perturbed measures $\hat{\mu}$ and $\mu^{(\alpha)}$ so that*

$$\begin{aligned} \hat{J}(z) &= \frac{J(z)}{(1-\overline{\alpha})J(z) + \overline{\alpha}} \\ C_\alpha(z) &= \frac{C(z)}{\alpha + (\alpha - 1)zC(z)}. \end{aligned}$$

Then $\hat{\mu}$, $\mu^{(\alpha)}$, and the Alexandrov measure $\mu_{\overline{\alpha}}$ are the same.

From our definitions above, it is obvious that for all $z \in \mathbb{D}$ we have

$$F(z) = J(z) + zC(z).$$

Using Theorems 4.5.1 and 4.5.2, we conclude that if $G(e^{i\gamma}) < \infty$ then $F(e^{i\gamma})$ is pure imaginary and $J(e^{i\gamma})$ has real part equal to $\frac{1}{2}$. It must then be the case that

$$D(e^{i\gamma}) \equiv \lim_{r\uparrow 1}(re^{i\gamma})C(re^{i\gamma}) = -\frac{1}{2} + iw$$

for some $w \in \mathbb{R}$. With this in mind, we have the following.

Proposition 4.6.2. *Let μ be a nontrivial probability measure on $\partial\mathbb{D}$. If $G(e^{i\gamma}) < \infty$ then $\lim_{r\uparrow 1} C(re^{i\gamma})$ exists.*

Proof. From the previous discussion we have that

$$\begin{aligned}
& \lim_{r\uparrow 1}(r\cos(\gamma) + ir\sin(\gamma))(\mathrm{Re}C(re^{i\gamma})) + i\mathrm{Im}C(re^{i\gamma}))) = -\frac{1}{2} + iw \\
\Rightarrow\ & \lim_{r\uparrow 1} r(\cos(\gamma)\mathrm{Re}C(re^{i\gamma})) - \sin(\gamma)\mathrm{Im}C(re^{i\gamma}))) = -\frac{1}{2} \\
\Rightarrow\ & \lim_{r\uparrow 1}(\cos(\gamma)\mathrm{Re}C(re^{i\gamma})) - \sin(\gamma)\mathrm{Im}C(re^{i\gamma}))) = -\frac{1}{2}.
\end{aligned}$$

By similar reasoning we have

$$\lim_{r\uparrow 1}(\cos(\gamma)\mathrm{Im}C(re^{i\gamma})) + \sin(\gamma)\mathrm{Re}C(re^{i\gamma}))) = w$$

and w is given by

$$w = \mathrm{Im}F(e^{i\gamma})) - \mathrm{Im}J(e^{i\gamma})) = F(e^{i\gamma}) - (J(e^{i\gamma}) - \frac{1}{2})$$

and so we can think of it as being fixed since $F(e^{i\gamma})$ and $J(e^{i\gamma})$ are known to exist by Theorems 4.5.1 and 4.5.2. A simple calculation then gives (assuming $\cos(\gamma) \neq 0$)

$$\begin{aligned}
\mathrm{Re}C(e^{i\gamma})) \equiv \lim_{r\uparrow 1}\mathrm{Re}C(re^{i\gamma})) &= \frac{\sin^2(\gamma) + w\sin(2\gamma) - 1}{2\cos(\gamma)} \\
\mathrm{Im}(C(e^{i\gamma})) \equiv \lim_{r\uparrow 1}\mathrm{Im}C(re^{i\gamma})) &= w\cos(\gamma) + \frac{\sin(\gamma)}{2}.
\end{aligned}$$

□

Chapter 5

Spectral Properties of Random Joye Matrices

5.1 Semi-infinite Joye Matrices

Let us define what we mean by a semi-infinite Joye matrix. A semi-infinite Joye matrix is a Joye matrix (as defined previously) that has been truncated in the upper left corner to produce a unitary semi-infinite matrix whose rows and columns can be indexed by $\mathbb{N}$. If we define the map $\theta_k : \Omega \to \mathbb{T}$ by

$$\theta_k : \Omega \to \mathbb{T}, \quad \text{s.t.} \quad \theta_k(\omega) = \omega_k, \quad k \in \mathbb{N}$$

then we can define the matrix D^+_ω by

$$D^+_\omega = \text{diag}\{e^{-i\theta_k(\omega)}\}.$$

Here we will define the semi-infinite matrix S^+ by

$$S^+ = \begin{pmatrix} -r_0 & r_1 t_0 & -t_0 t_1 & & & \\ t_0 & r_0 r_1 & -r_0 t_1 & & & \\ & r_2 t_1 & r_1 r_2 & r_3 t_2 & -t_2 t_3 & \\ & -t_1 t_2 & r_1 t_2 & r_2 r_3 & -r_2 t_3 & \\ & & & t_3 r_4 & r_3 r_4 & \\ & & & -t_3 t_4 & -r_3 t_4 & \ddots \end{pmatrix}$$

and the 2×4 block structure continues indefinitely. Notice that the values of r_j are also random. We will return to this issue later. Now we can define the semi-infinite Joye matrix U^+_ω by

$$U^+_\omega = D^+_\omega S^+.$$

Our first observation comes from [4] and says the following.

Proposition 5.1.1. *The matrix $-U^+_\omega$ is unitarily equivalent to a CMV matrix.*

The proof can be found in [4] and is very straightforward. Of key importance is the relationship between the Verblunsky coefficients and the randomness in the matrix U_ω^+. More precisely, if $-U_\omega^+$ is unitarily equivalent to a CMV matrix $\mathcal{C}$ in the way described in [4] then the Verblunsky coefficients of $\mathcal{C}$ are given by

$$\alpha_j = r_j e^{i\eta_j} \quad , \quad \eta_j = \sum_{k=0}^{j} \theta_k(\omega).$$

Conversely, if we are given the Verblunsky coefficients of $\mathcal{C}$, then we can derive the matrix elements of the matrix $-U_\omega^+$ by

$$r_j = |\alpha_j| \quad , \quad \theta_j = \eta_j - \eta_{j-1}.$$

This Proposition is very relevant for our purposes. First note that if $z \in \mathbb{C}$ is an eigenvalue of a matrix M then $-z$ is an eigenvalue of the matrix $-M$. The unitary equivalence mentioned above shows us that $-U_\omega^+$ shares its spectrum with a particular CMV matrix. Therefore, we can use results and methods concerning CMV matrices (such as orthogonal polynomials) to extract information about the spectrum of our matrix U_ω^+.

Recall also from [13] that the distribution of eigenvalues of a random CMV matrix can be determined by studying the distribution of the orthogonal polynomials that obey the recurrence relation

$$\Phi_{k+1}(z) = z\Phi_k(z) - \overline{\alpha}_k \Phi_k^*(z), \quad k \geq 0, \quad \Phi_0 = 1.$$

This will be the key to many of our results in the following sections.

5.2 Geronimus Polynomials

In this section, we will examine one particular case of how the spectrum of a semi-infinite Joye matrix can be deduced from the spectrum of a corresponding CMV matrix. We start with this example because it is by far the simplest. Recall that the Geronimus polynomials are the orthogonal polynomials obtained from the above recurrence relation when $\alpha_j = \alpha_k$ for all $j, k \geq 0$, that is, all of the α's are of the form $re^{i\theta_0}$ for some particular $0 < r < 1$ and $\theta_0 \in \mathbb{T}$. From our discussion in Section 5.1, the CMV matrix corresponding to a sequence of identical α's of the form $re^{i\theta_0}$ is unitarily equivalent to the matrix $-U_\omega^+$ given by $r = r_0 = r_1 = r_2 = \ldots$ and $0 = \theta_1 = \theta_2 = \theta_3 = \ldots$. From this we can deduce the following theorem.

Theorem 5.2.1. *Let U_ω^+ be a semi-infinite Joye matrix with matrix elements given by*

$$r = r_0 = r_1 = r_2 = \ldots, \quad 0 < r < 1$$
$$0 = \theta_1 = \theta_2 = \theta_3 = \ldots$$

and $\theta_0 \in \mathbb{T}$*. Define the parameters* α*,* β*, and* $\theta_{|\alpha|}$ *by*

$$\begin{aligned} \alpha &= re^{i\theta_0} \\ 1+\overline{\alpha} &= |1+\overline{\alpha}|e^{i\beta/2} \\ \theta_{|\alpha|} &= 2\arcsin(|\alpha|). \end{aligned}$$

The matrix $-U_\omega^+$ *has a.c. spectrum given by* $d\mu_{ac} = w(\theta)d\theta$ *where*

$$w(\theta) = \begin{cases} \frac{1}{|1+\alpha|}\frac{\sqrt{\cos^2(\theta_{|\alpha|}/2)-\cos^2(\theta/2)}}{\sin((\theta-\beta)/2)} & \text{if } \theta \in (\theta_{|\alpha|}, 2\pi - \theta_{|\alpha|}) \\ 0 & \text{otherwise.} \end{cases}$$

Also, $-U_\omega^+$ *has singular spectrum given by*

$$d\mu_s = \begin{cases} 0 & \text{if } |\alpha + \frac{1}{2}| \leq \frac{1}{2} \\ \frac{2}{|1+\alpha|^2}(|\alpha+\frac{1}{2}|^2 - \frac{1}{4})\delta_{\theta,\beta} & \text{otherwise.} \end{cases}$$

Proof. The matrix $-U_\omega^+$ is unitarily equivalent to the CMV matrix with Verblunsky coefficients given by $\alpha_j = re^{i\theta_0}$ for all $j \geq 0$. The spectrum of this CMV matrix (corresponding to Geronimus polynomials) is given explicitly in [9] (page 87) and is exactly as described above in the statement of the theorem. Since the matrices are unitarily equivalent, they have the same spectrum. The result follows. ☐

We will re-use the method we just employed several times throughout the remainder of this paper. We will start with a set of (possibly random) values for r_j and θ_j and translate these into values for α_j for the corresponding unitarily equivalent CMV matrix. We will then apply known results or methods concerning the spectrum of the CMV matrix and make a conclusion about the spectrum of the original semi-infinite Joye matrix.

5.3 The Uniform Distribution

In this section, we will describe a way in which the values of r_j and θ_j in the matrix U_ω^+ can be selected randomly so that the spectrum is almost surely pure point and the eigenvalues almost surely exhibit Poisson statistics. We start with the following lemma.

Lemma 5.3.1. *If* $x_1, x_2 \in [0,1]$ *are independent uniformly distributed random variables then so is* $y = x_1 + x_2 \bmod 1$.

Proof. Consider the expression $\mathbb{P}(y \in [\gamma, \gamma+\epsilon))$ for any $\gamma \in [0,1]$ and $\epsilon > 0$. This is given by

$$\begin{aligned}
&\mathbb{P}(y \in [\gamma, \gamma+\epsilon)) \\
= &\int_0^\gamma \gamma+\epsilon-x-(\gamma-x)dx + \int_{\gamma+\epsilon}^1 1+\gamma+\epsilon-x-(1+\gamma-x)dx \\
+ &\int_\gamma^{\gamma+\epsilon} (\gamma+\epsilon-x)dx + \int_\gamma^{\gamma+\epsilon} 1-(1+\gamma-x)dx \\
= &\int_0^\gamma \epsilon dx + \int_{\gamma+\epsilon}^1 \epsilon dx + \left((\gamma+\epsilon)x - \frac{x^2}{2}\right)|_\gamma^{\gamma+\epsilon} + \left(\frac{x^2}{2} - \gamma x\right)|_\gamma^{\gamma+\epsilon} \\
= &\epsilon(\gamma+1-(\gamma+\epsilon)) + \frac{(\gamma+\epsilon)^2}{2} - \frac{(\gamma)^2}{2} - \gamma\epsilon + \frac{\epsilon^2}{2} = \epsilon.
\end{aligned}$$

Therefore, the probability of finding y in an interval of length ϵ is ϵ so y is uniformly distributed on $[0, 1]$ as desired. □

From this, we get the following two Corollaries.

Corollary 5.3.2. *If $x_1, x_2 \in \mathbb{T}$ are independent uniformly distributed random variables then so is $y = x_1 + x_2 \bmod 2\pi$.*

Proof. The size of the interval does not effect the distribution. This is just a rescaling of Lemma 5.3.1. □

Corollary 5.3.3. *If $x_1, x_2, \ldots, x_n \in \mathbb{T}$ are independent uniformly distributed random variables then so is $y = \sum_{j=1}^n x_j \bmod 2\pi$.*

Proof. This follows from Corollary 5.3.2 and induction on n. □

We are now ready for the main result of this section.

Theorem 5.3.4. *Let U_ω^+ by a semi-infinite Joye matrix with $\{r_j\}_{j\geq 0}$ be i.i.d. random variables distributed according to the distribution $\frac{2x}{R^2}$ on $[0, R] \subset [0, 1]$ and $\{\theta_j\}_{j\geq 0}$ be i.i.d random variables distributed uniformly on $\mathbb{T}$. Then the matrix U_ω^+ has pure point spectrum almost surely and the eigenvalues of U_ω^+ exhibit Poisson statistics almost surely.*

Notice that the distribution of the r_j as given is already normalized.

Proof. Let us start by defining the sequence α_j by

$$\alpha_j = r_j e^{i\eta_j} \quad , \quad \eta_j = \sum_{k=0}^{j} \theta_k.$$

Clearly $\alpha_j \in \mathbb{D}$ for all $j \geq 0$. Consider the distribution of the α_j. By Corollary 5.3.3, the distribution of the parameters η_j is uniform in $\mathbb{T}$ so the distribution of the α_j is rotationally invariant. Now, for any $s \leq R$, the probability that $|\alpha_j| = r_j < s$ is given by

$$\int_0^s \frac{2x}{R^2} dx = \frac{s^2}{R^2}.$$

Notice also that the ratio of the disk of radius s to the disk of radius R is $\frac{s^2}{R^2}$. Therefore, the radial distribution of the α_j's is uniform, in that it is constant for $0 \leq r \leq R$. We conclude that if we select r_j and θ_j randomly as stated in the theorem, then the resulting matrix $-U_\omega^+$ is unitarily equivalent to a CMV matrix $\mathcal{C}_1$ with Verblunsky coefficients chosen randomly from the uniform distribution on the disk of radius R.

We know from [13] that the spectrum of a CMV matrix is almost surely pure point when the Verblunsky coefficients are chosen randomly from the disk of radius R and that the eigenvalues almost surely exhibit Poisson statistics. Therefore we conclude that $\mathcal{C}_1$ almost surely has pure point spectrum and the eigenvalues almost surely exhibit Poisson statistics and the same is true for $-U_\omega^+$. It is then trivial to reach the desired conclusion for U_ω^+. □

Chapter 6

Localization of Eigenfunctions

6.1 Aizenman's Theorem

In this section we would like to extend the main result presented in [3]. In that paper, Joye proves that if r is close to 1 (where "close to" depends on the distribution of the parameters θ_j) then there exists Aizenman-Molchanov bounds for the fractional moments of the expected value of the matrix elements of the resolvent (see Theorem 4.1.1 above). That is, if U_ω^+ is the random unitary matrix in question then there exist constants $K(s) > 0$ and $\gamma(s) > 0$ such that

$$\mathbb{E}(|\langle j|U_\omega^+(U_\omega^+ - z)^{-1}k\rangle|^s) \leq K(s)e^{-\gamma(s)|j-k|}$$

where the strength of the theorem comes from the fact that the constants $K(s)$ and $\gamma(s)$ are independent of $z \in \mathbb{D}$. We would like to adapt the proof in [11] to conclude that Aizenman's Theorem applies also to semi-infinite Joye matrices. This will allow us to prove that the eigenfunctions of the matrix U_ω^+ are exponentially localized. First, we need some notation and terminology. The following two definitions are taken from [11].

Definition 3. [11] *Let $\alpha \in \mathbb{D}^\infty$ be such that the collection $\{\alpha_j\}_{j\geq 0}$ are i.i.d. random variables distributed in $\mathbb{D}$. For $\lambda \in \partial\mathbb{D}$, define the operator $T_{n,\lambda} : \mathbb{D}^\infty \to \mathbb{D}^\infty$ by*

$$T_{n,\lambda} = \begin{cases} \alpha_j & \text{if } j < n \\ \lambda\alpha_j & \text{otherwise.} \end{cases}$$

Definition 4. [11] *We will denote by $D(\lambda^k(\lambda^{-1})^l(1\lambda)^\infty)$ the diagonal semi-infinite matrix with k λ's, l (λ^{-1})'s, and then alternating 1 and λ along the diagonal to infinity.*

With these definitions, Simon has the following theorem, where $\mathcal{C}(\alpha)$ is the CMV matrix corresponding to the sequence $\{\alpha_n\}_{n\geq 0}$.

Theorem 6.1.1. [11] *For $\lambda \in \partial\mathbb{D}$, define U_n for $n \in \mathbb{N}_0$ by*

$$\begin{aligned} U_{2k-1} &= D(1^{2k}(1\lambda)^\infty) \\ U_{2k} &= D(\lambda^{2k}(1\lambda)^\infty). \end{aligned}$$

Then

$$U_n \mathcal{C}(T_{n,\lambda^{-1}}(\alpha))U_n^{-1} = \mathcal{C}(\alpha)\Delta_n(\lambda)$$

where

$$\Delta_n(\lambda) = D(1^n \lambda 1^\infty).$$

Using Theorem 6.1.1, it follows immediately that

$$\left|[\mathcal{C}(T_{k,\lambda^{-1}}(\alpha))^n]_{km}\right| = \left|[(\mathcal{C}(\alpha)\Delta_k(\lambda))^n]_{km}\right|$$

for any k and m. We would like to draw a similar conclusion for semi-infinite Joye matrices. To help us do this, we have the following definition.

Definition 5. *Let $\omega \in \mathbb{T}^\infty$ be such that the collection $\{\omega_j\}_{j\geq 0}$ are i.i.d. random variables distributed in $\mathbb{T}$. For $\lambda \in \mathbb{T}$, define the operator $S_{n,\lambda} : \mathbb{T}^\infty \to \mathbb{T}^\infty$ by*

$$S_{n,\lambda} = \begin{cases} \lambda + \omega_j & \text{if } j = n \\ \omega_j & \text{otherwise.} \end{cases}$$

Recall that the unitary equivalence between a semi-infinite Joye matrix and its corresponding CMV matrix is given by the sequence of Verblunsky coefficients satisfying

$$\alpha_j = r_j e^{i(\theta_0+\theta_1+\cdots+\theta_j)}.$$

Therefore, the perturbation $S_{n,\lambda}$ of a semi-infinite Joye matrix induces the perturbation $T_{n,\lambda}$ in the corresponding CMV matrix.

Now we can prove an important preliminary result. Notice that this is an effective analog of Theorem 6.1.1 from [11], but the proof is much shorter.

Theorem 6.1.2. *For $\lambda \in \mathbb{T}$, we have*

$$\left|[U^+(S_{k,\lambda}(\omega))^n]_{km}\right| = \left|[(U^+(\omega)\Delta_k(e^{i\lambda}))^n]_{km}\right|$$

for any $n, k, m \in \mathbb{N}$.

Proof. Observe that

$$U^+(S_{k,\lambda}(\omega)) = \Delta_k(e^{i\lambda})U^+(\omega).$$

Since $e^{i\lambda} \in \partial\mathbb{D}$, the conclusion clearly follows for $n = 1$.

Now, for $n > 1$ consider

$$\begin{aligned} \left|[(U^+(\omega)\Delta_k(e^{i\lambda}))^n]_{km}\right| &= \left|[U^+(\omega)(\Delta_k(e^{i\lambda})U^+(\omega))^{n-1}\Delta_k(e^{i\lambda})]_{km}\right| \\ &= \left|[(U^+(\omega)\Delta_k(e^{i\lambda}))^{n-1}U^+(\omega)]_{km}\right| \end{aligned}$$

and similarly

$$\begin{aligned} \left|[(\Delta_k(e^{i\lambda})U^+(\omega))^n]_{km}\right| &= \left|[\Delta_k(e^{i\lambda})(U^+(\omega)\Delta_k(e^{i\lambda}))^{n-1}U^+(\omega)]_{km}\right| \\ &= \left|[(U^+(\omega)\Delta_k(e^{i\lambda}))^{n-1}U^+(\omega)]_{km}\right| \end{aligned}$$

and the desired conclusion follows immediately. ∎

We will prove Aizenman's Theorem for semi-infinite Joye matrices whenever $\tau(\theta)$ is an absolutely continuous distribution on $\mathbb{T}$ satisfying the following conditions

1. $\tau(\theta) > 0$ for all $\theta \in \mathbb{T}$,
2. $\max_{\theta\in\mathbb{T}} \tau(\theta)$ and $\min_{\theta\in\mathbb{T}} \tau(\theta)$ exist.

It should be noted that these conditions are not very strong. Note also that these conditions imply that if we define β so that

$$\beta = \frac{\sup_{\theta\in\mathbb{T}} \tau(\theta)}{\inf_{\theta\in\mathbb{T}} \tau(\theta)}$$

then for any $\varphi \in \mathbb{T}$ we have

$$\beta^{-1}\tau(\theta+\varphi) \leq \tau(\theta) \leq \beta\tau(\theta+\varphi).$$

This fact is essentially what allows us to prove the desired result.

Shown above (Figure 7.1) is a plot that shows the absolute value of the $(7,9)$ entry of two 15×15 Joye matrices raised to all powers $n \leq 500$. The blue plot is the absolute value when the phases are chosen from the uniform distribution on $[0, 5\pi)$ and the red plot is the absolute value when θ_7 has been perturbed slightly (i.e. θ_7 has been chosen from the uniform distribution on $[\varphi, 5\pi+\varphi)$ for some $\varphi \in \mathbb{T}$). We see that although the plots do not have the same supremum, the quantities being plotted are of the same order of magnitude and we will see below that if we take the expected value of the suprema of these quantities they will not differ by more than a factor of β.

Now let us define a function $f_{kl} : \mathbb{T}^\infty \to \mathbb{R}_{\geq 0}$ by

$$f_{kl}(\lambda) = \sup_n |[U^+(\lambda)^n]_{kl}|$$

for any $\lambda \in \mathbb{T}^\infty$. Since all powers of a unitary matrix are unitary, we get that $f_{kl}(\lambda) \leq 1$ for all $\lambda \in \mathbb{T}^\infty$ and all $k, l \in \mathbb{N}$. It follows that $f_{kl}(\lambda)$ is an integrable function for all $k, l \in \mathbb{N}$. First we will show that f_{kl} is measurable for all $k, l \in \mathbb{N}$.

Proposition 6.1.1. *The function $f_{kl} : \mathbb{T}^\infty \to \mathbb{R}_{\geq 0}$ given by*

$$f_{kl}(\lambda) = \sup_n |[U^+(\lambda)^n]_{kl}|$$

is measurable for all $k, l \in \mathbb{N}$.

Proof. Let us define a sequence of functions $f_{kl,n} : \mathbb{T}^\infty \to \mathbb{R}_{\geq 0}$ by

$$f_{kl,n}(\lambda) = |[U^+(\lambda)^n]_{kl}|.$$

It is clear that $f_{kl,n}(\lambda)$ is measurable for all $k, l, n \in \mathbb{N}$. Since $f_{kl}(\lambda) = \sup_n f_{kl,n}(\lambda)$ then for any $\lambda \in \mathbb{R}$ we get that

$$f_{kl}^{-1}((-\infty,\lambda)) = \bigcap_{n\geq 1} f_{kl,n}^{-1}((-\infty,\lambda)),$$

which is certainly a measurable set. □

This functional form of $\sup_n |[U^+(\omega)^n]_{kl}|$ leads us to the following lemma.

Lemma 6.1.3. *Let $\omega \in \mathbb{T}^\infty$ be such that*

$$\omega = (\theta_0, \theta_1, \theta_2, \dots)$$

where the distribution of each random variable θ_j is $\tau(\theta)$ satisfying the conditions mentioned earlier. Similarly, let $\omega' \in \mathbb{T}^\infty$ be such that

$$\omega' = (\phi_0, \phi_1, \phi_2, \dots)$$

where the distribution of each random variable ϕ_j for $j \neq k$ is $\tau(\phi)$ satisfying the conditions mentioned earlier and the distribution of the random variable ϕ_k is $\tau(\phi - \varphi)$ for some $\varphi \in \mathbb{T}$. Then for any $k, l \geq 0$

$$\mathbb{E}(\sup_n |[U^+(\omega)^n]_{kl}|) \leq \beta \mathbb{E}(\sup_n |[U^+(\omega')^n]_{kl}|).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}(\sup_n |[U^+(\omega)^n]_{kl}|) &= \mathbb{E}(f_{kl}(\omega)) \\
&= \cdots \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} f_{kl}(\lambda) \tau(\lambda_k) d\lambda_k \tau(\lambda_0) d\lambda_0 \tau(\lambda_1) d\lambda_1 \cdots \\
&\leq \beta \cdots \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} f_{kl}(\lambda) \tau(\lambda_k - \varphi) d\lambda_k \tau(\lambda_0) d\lambda_0 \tau(\lambda_1) d\lambda_1 \cdots \\
&= \beta \mathbb{E}(f_{kl}(\omega')) \\
&= \beta \mathbb{E}(\sup_n |[U^+(\omega')^n]_{kl}|)
\end{aligned}$$

as desired. □

This allows us to prove Aizenman's Theorem for semi-infinite Joye matrices when the distribution $\tau(\theta)$ of the random variables $\{\theta_j\}_{j \geq 0}$ satisfies the weak conditions mentioned previously.

Theorem 6.1.4. *Let $U^+(\omega)$ be a semi-infinite Joye matrix where the distribution of the random variables $\{\theta_j\}_{j \geq 0}$ is $\tau(\theta)$ satisfying the conditions mention above and the random variables $\{r_j\}_{j \geq 0}$ are such that*

$$\mathbb{E}(|\langle j | U^+(\omega)(U^+(\omega) - z)^{-1} k \rangle|^s) \leq K e^{-\gamma |j-k|}$$

for suitably chosen constants K and $\gamma > 0$. Then there exist constants K_1 and $\gamma_1 > 0$ such that

$$\mathbb{E}(\sup_n |[(U^+(\omega))^n]_{jk}|) \leq K_1 e^{-\gamma_1 |j-k|}.$$

The proof uses the same method used in [11].

Proof. Define $J_{km}(z)$ by

$$J_{km}(z) = [U^+(\omega)(U^+(\omega) - z)^{-1}]_{km}.$$

By Theorem 4.2 in [11], we conclude that

$$\int \frac{d\varphi}{2\pi} \sup_n |[(U^+(\omega)\Delta_k(e^{i\varphi}))^n]_{km}| \leq \left(\int |F_{km}(e^{i\theta})|^s \frac{d\theta}{2\pi} \right)^{1/(2-s)}$$

for $0 < s < 1$. Using Theorem 4.2.1 from Section 4.2, we conclude that

$$\int \frac{d\varphi}{2\pi} \sup_n |[(U^+(\omega)\Delta_k(e^{i\varphi}))^n]_{km}| \leq 2\left(\int |J_{km}(e^{i\theta})|^s \frac{d\theta}{2\pi} \right)^{1/(2-s)}$$

for $0 < s < 1$. Using Theorem 6.1.2 we get

$$\int \frac{d\varphi}{2\pi} \sup_n |[(\Delta_k(e^{i\varphi})U^+(\omega))^n]_{km}| \leq 2\left(\int |J_{km}(e^{i\theta})|^s \frac{d\theta}{2\pi} \right)^{1/(2-s)}.$$

Now we can take expectations of both sides. Notice that the matrix $\Delta_k(e^{i\varphi})U^+(\omega)$ is the matrix we would get if the random variables $\{\theta_1, \theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots\}$ were distributed according to the distribution $\tau(\theta)$ on $\mathbb{T}$ and the random variable θ_k was distributed according to the distribution $\tau(\theta - \varphi)$ on $\mathbb{T}$. Therefore, by Lemma 6.1.3 we conclude that

$$\frac{1}{\beta}\mathbb{E}(\sup_n |[(U^+(\omega))^n]_{km}|) \leq \mathbb{E}(\sup_n |[(\Delta_k(e^{i\varphi})U^+(\omega))^n]_{km}|).$$

Now, as in [11], we can use Hölder's inequality to bring the $\mathbb{E}$ inside the $(\cdot)^{1/(2-p)}$ and conclude that

$$\mathbb{E}(\sup_n |[(U^+(\omega))^n]_{km}|) \leq 2\beta\left(\int |J_{km}(e^{i\theta})|^s \frac{d\theta}{2\pi} \right)^{1/(2-s)}.$$

The desired conclusion follows immediately. □

This result is very similar to the one presented in [11], but it has two important differences. First, in the statement of the main theorem in [11], the non-trivial condition that must be satisfied is a condition on the spectral measures associated with the CMV matrices corresponding to the sequences of Verblunsky coefficients α and $T_{n,\lambda}(\alpha)$ (see Definition 3). The condition we use is a statement about the angular distribution of the parameters in the semi-infinite Joye matrix and is easily verifiable whenever the randomness of the matrix is given explicitly. The second main difference is that the proof presented here easily generalizes to the case when the phases are not identically distributed, but still satisfy the conditions on the distribution $\tau(\theta)$ above (with the same β).

Let us consider what the above theorem means for CMV matrices. Recall that the matrix $-U^+(\omega)$ is unitarily equivalent to a CMV matrix with Verblunsky coefficients given by

$$\alpha_j = r_j e^{i(\theta_0 + \cdots + \theta_j)}.$$

We would like to understand the distribution of α_j induced by the distribution $\tau(\theta)$ satisfying the conditions of Theorem 6.1.4. To this end we have the following Proposition.

Proposition 6.1.2. *Let $\tau(\theta)$ be a probability distribution on $\mathbb{T}$ satisfying the conditions of Theorem 6.1.4 and satisfying $\beta < 2$. If θ_i is chosen randomly from the distribution $\tau(\theta)$ as in Theorem 6.1.4 then the distribution of α_j defined as*

$$\alpha_j = e^{i(\theta_0 + \cdots + \theta_j)}$$

converges to the uniform distribution on $\mathbb{T}$ as $j \to \infty$.

Notice that we have $|\alpha_j| = 1$ for all $j \geq 0$. The proof easily generalizes for any norm.

Proof. Let us denote by $\tau_j(\theta)$ the distribution of α_j (clearly $\tau_0(\theta) = \tau(\theta)$). Let us write the distribution $\tau_0(\theta)$ as

$$\tau_0(\theta) = \frac{1}{2\pi} + \epsilon_0(\theta) \quad \text{where} \quad \int_{\mathbb{T}} \epsilon_0(\theta) d\theta = 0.$$

Since $\beta < 2$ it follows that $0 < \tau_0(\theta < \frac{1}{\pi}$. Since $\tau_0(\theta)$ must achieve a maximum and a minimum on $\mathbb{T}$ we can find a $\delta \in (0,1)$ such that

$$|\epsilon_0(\theta)| < \frac{\delta}{2\pi}.$$

For any $\gamma \in \mathbb{T}$ we can write

$$\begin{aligned} |\tau_1(\gamma)| &= |\int_0^\gamma (\frac{1}{2\pi} + \epsilon_0(x))(\frac{1}{2\pi} + \epsilon_0(\gamma - x))dx \\ &+ \int_\gamma^{2\pi} (\frac{1}{2\pi} + \epsilon_0(x))(\frac{1}{2\pi} + \epsilon_0(2\pi + \gamma - x))dx| \\ &= |\int_0^{2\pi} \frac{1}{4\pi^2} dx + \int_0^{2\pi} \frac{\epsilon_0(x)}{2\pi} dx + \int_0^{2\pi} \frac{\epsilon_0(\gamma - x)}{2\pi} dx \\ &+ \int_0^\gamma \epsilon_0(x)\epsilon_0(\gamma - x)dx + \int_\gamma^{2\pi} \epsilon_0(x)\epsilon_0(2\pi + \gamma - x)dx| \\ &< \frac{1}{2\pi} + 0 + 0 + \frac{\delta^2}{2\pi}. \end{aligned}$$

Similar reasoning shows $|\tau_1(\gamma)| > \frac{1}{2\pi} - \frac{\delta^2}{2\pi}$.

Repeating this process to find $\tau_2(\theta)$ shows that the error term is smaller than $\frac{\delta^3}{2\pi}$. Since $\delta \in (0,1)$, as we repeat this process infinitely, the error terms converge to zero so the distribution must converge to the uniform as desired. □

Now that we have Aizenman's Theorem, we can use the proof of Theorem 2.2 in [13] to obtain the following theorem for semi-infinite Joye-Matrices..

Theorem 6.1.5. *If $U^+(\omega)$ is a semi-infinite Joye matrix randomized so that the conclusion of Theorem 6.1.4 holds then the eigenfunctions of the random matrix $(U^+(\omega))^{(n)}$ (the matrix $U^+(\omega)$ but $|r_{n-1}| = 1$) are exponentially localized with probability 1, that is, exponentially small outside sets of size proportional to $\ln(n)$. This means that there exists a constant $D_2 > 0$ and for almost every $\omega \in \Omega$, there exists a constant $C_\omega > 0$ such that for any eigenfunction $\varphi_\omega^{(n)}$, there exists a point $m(\varphi_\omega^{(n)})$ (with $1 \leq m(\varphi_\omega^{(n)}) \leq n$) with the property that for any m such that $|m - m(\varphi_\omega^{(n)})| \geq D_2 \ln(n+1)$ we have*

$$|\varphi_\omega^{(n)}(m)| \leq C_\omega e^{-(4/D_2)|m-m(\varphi_\omega^{(n)})|}.$$

Outline of Proof. First, we consider expressions of the form

$$\int_\Omega \left(\sup_n \left| \left(\delta_k, (U^+(\omega))^n \delta_l \right) \right| \right) d\mathbb{P}(\omega).$$

By Theorem 6.1.4, we conclude that this integral decays exponentially as $|k-l| \to \infty$. Then, as in [13], we conclude that

$$\sup_n \left| \left(\delta_k, (U^+(\omega))^n \delta_l \right) \right| \leq D_\omega (1+n)^6 e^{-D_0|k-l|}$$

for appropriately chosen constants $D_\omega, D_0 > 0$. It is then straightforward to show that if $|k-l| \geq \frac{12}{D_0} \ln(n+1)$ then we can get a stronger condition, namely

$$\sup_n \left| \left(\delta_k, (U^+(\omega))^n \delta_l \right) \right| \leq C_\omega e^{-\frac{D_0}{2}|k-l|}.$$

The rest of the proof is done exactly as described in Section 3.2. Thus, we have completed the second step (as outlined in Chapter 2) towards proving the existence of Poisson statistics for the distribution of the eigenvalues of the matrix $U^+(\omega)$.

Chapter 7

Further Work

The are several questions that could still be explored given the above results. The first is obviously to classify the asymptotic distribution of the eigenvalues of the random semi-infite Joye matrices satisfying the conditions of Theorem 6.1.4 so that Aizenman's Theorem holds true. Numerical analysis and Mathematica plots indicate that the asymptotic distribution of the eigenvalues of random semi-infinite Joye matrices is almost surely Poisson. However, the proof will require some additional work to cover the broad class of distributions for which we now know Aizenman's Theorem holds true.

A second topic to be explored is to understand the picture presented in Section 6.1. If we generated more of these plots in the same way but re-randomizing the matrix each time, we would find that nearly all of them show some sort of easily identifiable periodic behavior. If we could understand these plots, it could be helpful in finding a proof of Aizenman's Theorem that does not depend on the Aizenman-Molchanov bounds on the fractional moments, which would be a very big result.

Finally, it would be interesting to interpret the results of Chapter 6 in terms of CMV matrices. We have already stated that if the distribution $\tau(\theta)$ is close to being uniform, then the distribution $\nu_n(\alpha)$ of the Verblunsky coefficient α_n converges to a rotation invariant distribution as $n \to \infty$. It would be interesting to find a specific set of distributions $\{\tau_n(\theta)\}_{n\geq 0}$ such that the distribution of ω_j is $\tau_j(\theta)$ and the asymptotic distribution of the corresponding Verblunsky coefficients is not rotation invariant.

Appendix A

Splitting the Integral

The following lemma resulted from our approach in Chapter 3. It is not explicitly cited in any proof, but the idea behind this argument is worth articulating.

Lemma .0.6. *Let $f, g : \mathbb{C} \to \mathbb{R}$ be functions defined a.e. on some simply connected compact set $K \subset \mathbb{C}$ satisfying*

1. $f(z), g(z) \geq 0$ *for a.e.* $z \in K$,
2. *There exist disjoint sets* $X, Y \subseteq K$ *such that* $X \cup Y = K$, $f|X$ *is defined everywhere and bounded,* $g|Y$ *is defined everywhere and bounded,*
3. $f, g \in \mathcal{L}(K, d\lambda(z))$.

Then $fg \in \mathcal{L}(K, d\lambda(z))$.

Proof. Let us say that $f|X$ is bounded by some constant C_1 and $g|Y$ is bounded by some constant C_2. Then

$$\begin{aligned} \int_K fg d\lambda(z) &= \int_X fg d\lambda(z) + \int_Y fg d\lambda(z) \\ &\leq C_1 \int_X g d\lambda(z) + C_2 \int_Y f d\lambda(z) \\ &\leq C_1 \int_K g d\lambda(z) + C_2 \int_K f d\lambda(z) < \infty \end{aligned}$$

as desired. □

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