

WCMC Math Challenge Questions - Summer 2015

Instructions. Try as many of the following questions as you can. There is no time limit. We are more interested in how you approach the questions and how you communicate your reasoning than in how many correct answers you obtain. For each solution you submit, please include a clear and complete explanation of your answer. Send any questions you have to apacelli@williams.edu.

1. There's a small village at the base of a mountain with a population of 8000. There are at least 2 people in the village that do not have dogs. Also, given any three people in the village, at least one of the three has a dog. Do we have enough information to determine exactly how many people in the village have dogs? Explain why not, or find the exact number. Explain your reasoning.

2. Let m and n be positive integers. If m has a digits and n has b digits, what are the possibilities for the number of digits that the product mn has? Explain your answer completely.

3. You are blindfolded, and on the table in front of you are a number of shiny smooth discs. They are each about the size of a quarter; one side is black and the other side white. You can certainly determine how many discs there are altogether, but you can't tell whether a given disc is black side up or white side up. Your friend Ed tells you how many are black side up. Your challenge is to separate the discs into two collections, flipping over whichever discs you like, so that each collection has the same number of black side up discs. Explain how you complete the challenge, and how you know you're successful.

4. Find all integers M that satisfy all of the following properties. Explain your answer fully.

(i) If 3 divides M , then $20 \leq M \leq 40$.

(ii) If 9 does not divide M , then $20 \leq M \leq 30$.

(iii) 7 divides M .

5. There are 100 doors, and 100 people lined up in a row. All the doors are closed. The first person goes through and opens all the doors. The second person goes through and changes the position (open to closed or closed to open) of every second door. The third person goes through and changes the position of every third door. And so on. After all 100 people have gone through the doors, which doors are open and which are closed? Explain fully.

6. There's a box of five hats: two blue and three white. Andy, Kate, and James each place a hat on his or her head, while blindfolded. One by one, each child removes his blindfold and (without using a mirror) gets one opportunity to guess the color of the hat on his own

head. If any of the three guesses correctly, everyone gets to go to the park! Andy, Kate, and James are each very logical, and know that the others are as well.

First, James removes his blindfold. He sees the hats that the others are wearing, but admits that he is unable to discern his own hat color.

Next, Kate removes her blindfold, and sadly reveals that she too is not able to determine the color of her own hat.

Finally, Andy pipes up and says “I can answer with my blindfold on! I know what color hat I am wearing.”

What color is Andy’s hat, and how does he know? Explain fully.

7. Andy encounters a strange island, where every creature has either green, purple, or blue hair on his head. He’s told by a reliable source that those with green hair always tell the truth, those with purple hair always lie, and those with blue hair make statements that are alternately true and false (though the order of which statements are true and which are false is unknown). One day, there is a race on the island. After the race, Andy talks to the top three finishers (there were no ties), but each is wearing a very big hat which completely covers his or her hair. Each of the three islanders makes three statements.

A: 1. I won the race.

2. B was second.

3. C was third.

B: 1. I won the race.

2. I was well out in front the entire race!

3. C finished behind A and me.

C: 1. I won the race.

2. $5 + 7 = 12$.

3. B finished ahead of A.

Who won the race? And what color hair do islanders A, B, and C have? (Note: You cannot assume that each has a different hair color.) Explain your answers fully.