WCMC Math Challenge Questions - Summer 2014

Instructions. Try as many of the following questions as you can. There is no time limit. We are more interested in how you approach the questions and how you communicate your reasoning than in how many correct answers you obtain. For each solution you submit, please include a clear and complete explanation of your answer. Send any questions you have to apacelli@williams.edu.

1. One thousand politicians attended a recent convention. You know two facts: (1) At least one of the politicians is honest, and (2) Given any two politicians at the convention, at least one of those two is a liar. Do you have enough information to determine exactly how many of the politicians are honest? Explain why or why not, and give the exact number of honest politicians if possible.

2. If 29 bakery staffers can decorate 29 cupcakes in 29 minutes, then how many bakery staffers are required to decorate 87 cupcakes in 87 minutes?

3. John McClane and his friend Zeus are trying to stop the bomb that Simon Gruber set! It will go off in 3 minutes unless you can help them defuse it! Before you are four glasses. One glass of indiscernible size contains nine ounces of water. The other three glasses are empty and unlabeled, but Simon told you that they hold exactly five, four, and two ounces of liquid respectively. The bomb will go off in 3 minutes unless you divide the nine ounces of water into three glasses containing exactly three ounces each. Be careful not to spill any water because there's no more nearby (close enough to access within the time limit). Good luck! And be sure to clearly write up what you did and why it worked when you're done defusing the bomb.

4. 1. Find all integers M that satisfy the following properties. Explain your answer fully.

   (i) If 3 divides M, then $20 \leq M \leq 40$.

   (ii) If 9 does not divide M, then $20 \leq M \leq 30$.

   (iii) 7 divides M.

5. There are 100 doors, and 100 people lined up in a row. All the doors are closed. The first person goes through and opens all the doors. The second person goes through and changes the position (open to closed or closed to open) of every second door. The third person goes through and changes the position of every third door. And so on. After all 100 people have gone through the doors, which doors are open and which are closed? Explain fully.

6. There's a box of five hats: two blue and three white. Andy, Kate, and James (each very

smart and very logical) each place a hat on his or her head, while blindfolded. One by one, each child removes his blindfold and (without using a mirror) gets one opportunity to guess the color of the hat on his own head. If any of the three guesses correctly, everyone gets to go to the park!

First, James removes his blindfold. He sees the hats that the others are wearing, but admits that he is unable to discern his own hat color.

Next, Kate removes her blindfold, and sadly reveals that she too is not able to determine the color of her own hat.

Finally, Andy pipes up and says "I can answer with my blindfold on! I know what color hat I am wearing."

What color is Andy's hat, and how does he know?

7. Andy encounters a strange island, where every creature has either green, purple, or blue hair on his head. He's told by a reliable source that those with green hair always tell the truth, those with purple hair always lie, and those with blue hair make statements that are alternately true and false (though the order of which statements are true and which are false is unknown). One day, there is a race on the island. After the race, Andy talks to the top three finishers (there were no ties), but each is wearing a very big hat which completely covers his or her hair. Each of the three islanders makes three statements.

A: 1. I won the race.

2. B was second.

3. C was third.

B: 1. I won the race.

2. I was well out in front the entire race!

3. C finished behind A and me.

C: 1. I won the race.

2. $5 + 7 = 12$.

3. B finished ahead of A.

Who won the race? And what color hair do islanders A, B, and C have? (Note: You cannot assume that each has a different hair color.) Explain your answers fully.